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Topics in Combinatorics IV, Solutions 16 (Week 16)

16.1. Let Δ be a root system, Π is a set of simple roots, $\alpha_i \in \Pi$.

(a) ($\star$) Show that $r_{\alpha_i}(\Delta^+ \setminus \alpha_i) = \Delta^+ \setminus \alpha_i$. In other words, r_{α_i} takes all positive roots except α to positive roots.
Hint: use Theorem 9.12.

(b) Let $w \in W$, $\alpha \in \Pi$. Denote $n(w) = \#\{\beta \in \Delta^+ \mid w\beta \in \Delta^-\}$, i.e. the number of positive roots taken by w to negative ones. Show that if $w\alpha \in \Delta^+$ then $n(wr_\alpha) = n(w)+1$, and if $w\alpha \in \Delta^-$ then $n(wr_\alpha) = n(w)-1$. In particular, $n(w) \le l(w)$.

(c) Let $s_1 \dots s_k$ be a reduced expression for w, where $s_i = r_{\alpha_i}$ are simple reflections. Show that if $n(w) < l(w)$ then there exist $i < j$ such that $s_i(s_{i+1} \dots s_{j-1})\alpha_j = \alpha_i$.

(d) Show that $n(w) = l(w)$ for every $w \in W$.

Solution:

(a) Let $\beta = \sum c_j\alpha_j \in \Delta^+$, i.e. $c_j \ge 0$. Since $\beta \ne \alpha_i$, there is $j \ne i$ such that $c_j > 0$. Then $r_{\alpha_i}(\beta) = \beta - c\alpha_i$ has a positive coefficient c_j before α_j (as it has not changed), and thus must be positive.

(b) Let $w\alpha \in \Delta^+$, and assume $wr_\alpha(\alpha') \in \Delta^-$ for some $\alpha' \in \Delta^+$. By (a), either $\alpha' = \alpha$ (and then $wr_\alpha(\alpha') = wr_\alpha(\alpha) = w(-\alpha) = -w\alpha \in \Delta^-$), or $\alpha'' := r_\alpha(\alpha') \in \Delta^+$, and thus $\alpha' = r_a(\alpha'')$, where $\alpha'' \in \Delta^+$ is taken by w to Δ^-. Therefore, $n(wr_\alpha) = n(w)+1$.
If $w\alpha \in \Delta^-$, then $wr_\alpha\alpha \in \Delta^+$, and, by the reasoning above, $n(w) = n((wr_\alpha)r_\alpha) = n(wr_\alpha)+1$.
As $n(ws) \le n(w)+1$ for any simple reflection s, we see that $n(w) \le l(w)$.

(c) Suppose that $n(w) < l(w)$. Applying (b), we see that there is some index j such that $n((s_1 \dots s_{j-1})s_j) \ne n(s_1 \dots s_{j-1}) + 1$, which implies that $(s_1 \dots s_{j-1})\alpha_j \in \Delta^-$. Now, find the maximal i such that $(s_{i+1} \dots s_{j-1})\alpha_j \in \Delta^+$ and $s_i(s_{i+1} \dots s_{j-1})\alpha_j \in \Delta^-$ (such i exists since $\alpha_j > 0$). By (a), $s_{i+1} \dots s_{j-1}\alpha_j = \alpha_i$.

(d) By (b), $n(w) \le l(w)$. If $n(w) < l(w)$ then, by (c), $s_{i+1} \dots s_{j-1}\alpha_j = \alpha_i$, which implies $(s_{i+1} \dots s_{j-1})s_j(s_{j-1} \dots s_{i+1}) = s_i$. Therefore, $s_is_{i+1} \dots s_{j-1}s_j = s_{i+1} \dots s_{j_1}$, which means that the word was not reduced, so we get a contradiction.

16.2. Let Δ be a root system. Show that the highest root $\tilde\alpha_0$ is always long, i.e. $(\tilde\alpha_0, \tilde\alpha_0) \ge (\alpha, \alpha)$ for any $\alpha \in \Delta$.

Solution:

Let C_0 be the Weyl chamber. Take any $\alpha \in \Delta$. Since C_0 is a fundamental domain for the action of the Weyl group W, there exists $w \in W$ such that $w(\alpha) = \beta \in \overline{C}_0$. Thus, we can assume $(\beta, \alpha_i) \ge 0$ for any i.

Now, since $\tilde\alpha_0$ is the highest root, we have $\tilde\alpha_0 - \beta = \sum c_i\alpha_i$ with all $c_i \ge 0$. This implies $(\tilde\alpha_0 - \beta, \beta) = (\sum c_i\alpha_i, \beta) = \sum c_i(\alpha_i, \beta) \ge 0$, and $(\tilde\alpha_0 - \beta, \tilde\alpha_0) = (\sum c_i\alpha_i, \tilde\alpha_0) = \sum c_i(\alpha_i, \tilde\alpha_0) \ge 0$. Therefore,

$$(\tilde\alpha_0, \tilde\alpha_0) \ge (\beta, \tilde\alpha_0) \ge (\beta, \beta) = (\alpha, \alpha)$$

16.3. Let (G, S) be a Coxeter system, let $T \subset S$, and let G_T be a standard parabolic subgroup (see HW 15.2). Define $G^T = \{g \in G \mid l(gt) > l(g) \forall t \in T\}$. Let $g \in G$.

(a) Let $u_0 \in gG_T$ be a coset representative of minimal possible length across the whole coset. Show that $u_0 \in G^T$ and $g = u_0 v_0$ for some $v_0 \in G_T$.

(b) Show that $l(g) = l(u_0) + l(v_0)$.

(c) Show that every $p \in gG_T$ can be written as $p = u_0 v$ for some $v \in G_T$ with $l(p) = l(u_0) + l(v)$.

(d) Show that u_0 is the unique element of gG_T of minimal length.

(e) Show that there is a unique $u \in G^T$ and a unique $v \in G_T$ such that $g = uv$.

Solution:

(a) Since $u_0 \in gG_T$, we have $u_0 = gv$ for $v \in G_T$, and thus $u_0 v_0 = g$ for $v_0 = v^{-1} \in G_T$. Further, for $t \in T$, $u_0 t = gvt = g(vt) \in gG_T$, so due to the minimality of length of u_0 we have $l(u_0 t) > l(u_0)$, and thus $u_0 \in G^T$.

(b) Let $s_1 \dots s_k$ and $t_1 \dots t_m$ be reduced expressions for u_0 and v_0 respectively. If $l(g) < l(u_0) + l(v_0)$, then the word $s_1 \dots s_k t_1 \dots t_m$ is not reduced, so, by the Deletion Condition, we can remove two letters. If at least one of them is s_i, then we get $s_1 \dots \hat{s}_i \dots s_k \sim_G gv'$ for $v' \in G_T$ in contradiction with the minimality of u_0. Therefore, $g \sim_G s_1 \dots s_k t_1 \dots \hat{t}_i \dots \hat{t}_j \dots t_m$, so $t_1 \dots \hat{t}_i \dots \hat{t}_j \dots t_m \sim_G u_0^{-1} g = v_0$, which contradicts the fact $t_1 \dots t_m$ is a reduced expression for v_0.

(c) This follows from the two steps above: we have only used that $g \in gG_T$.

(d) Due to (c), for every $u_0' \in gG_T$ we have $u_0' = u_0 v$ with $l(u_0') = l(u_0) + l(v)$. If u_0' is minimal, then $l(v) = 0$ and thus $u_0' = u_0$.

(e) Suppose there is $u \in G^T$ and $v \in G_T$ such that $g = uv$, $u \neq u_0$. Then $u = gv^{-1} \in gG_T$. By (c), we can write $u = u_0 v'$, let $v' \sim_G t_1 \dots t_m$ be a reduced expression. Then $l(ut_m) \leq l(u_0) + (m-1) < l(u)$, so we come to a contradiction.

16.4. $(\star)$ Let G be a finite Coxeter group, (G, S) is a Coxeter system.

(a) Show that there is a unique element g_0 of maximal length. What is its length?

(b) Write down g_0 for the group of type A_3.

Solution:

(a) Length of an element $g \in G$ is the number of reflections in its R-sequence appearing one time only, or, equivalently, the number of mirrors of reflections separating the initial chamber C_0 from gC_0. There is only one chamber separated from C_0 by mirrors of all reflections, this is $-C_0$. Therefore, $g_0 C_0 = -C_0$. In particular, $l(g_0)$ is the number of reflections in G.

(b) There are 6 reflections in A_3 (e.g., S_4 has six transpositions, or this can be seen by realizing A_3 as the symmetry group of a regular tetrahedron): these are

$$s_1, s_2, s_3, s_1 s_2 s_1, s_2 s_3 s_2, s_1 s_2 s_1 s_3 s_1 s_2 s_1 (= s_1 s_2 s_3 s_2 s_1).$$

We can arrange them as follows:

$$\begin{aligned}
&s_1, (s_1) s_2 (s_1), (s_1 s_2) s_3 (s_2 s_1), (s_1 s_2 s_3) s_2 (s_3 s_2 s_1) = s_1 s_3 s_1 = s_3,\\
&(s_1 s_2 s_3 s_2) s_1 (s_2 s_3 s_2 s_1) = s_1 s_3 s_2 s_3 s_1 s_3 s_2 s_3 s_1 = s_1 s_3 s_2 s_1 s_2 s_3 s_1 = s_1 s_3 s_1 s_2 s_1 s_3 s_1 = s_3 s_2 s_3,\\
&\qquad (s_1 s_2 s_3 s_2 s_1) s_2 (s_1 s_2 s_3 s_2 s_1) = s_1 s_2 s_3 s_1 s_3 s_2 s_1 = s_1 s_2 s_1 s_2 s_1 = s_2,
\end{aligned}$$

so $g_0 = s_1 s_2 s_3 s_2 s_1 s_2$. Of course, there are many ways to write down g_0 leading to distinct R-sequences.

Alternatively, one can construct directly the element taking C_0 to $-C_0$. We know that $W(A_3) = S_4$ acts on $\mathbb{R}^4 \cap H$ by permutations of coordinate axes, and that $C_0 = \{x \in \mathbb{R}^4 \mid x_1 > x_2 > x_3 > x_4\} \cap H$, where $H = \{x \in \mathbb{R}^4 \mid x_1 + x_2 + x_3 + x_4 = 0\}$. Then $-C_0$ will be given by inequalities $x_1 < x_2 < x_3 < x_4$, and thus $g_0 = \binom{1\,2\,3\,4}{4\,3\,2\,1} \in S_4$. We are left to write g_0 down as a product of transpositions $s_i = (i\, i+1)$. Observe that $g_0 = (14)(23)$, so we just need to write down (14) as a product of simple reflections. Clearly, $(14) = (13)(34)(13) = ((12)(23)(12))(34)((12)(23)(12)) = (12)(23)(34)(23)(12)$, so $g_0 = s_1 s_2 s_3 s_2 s_1 s_2$ works indeed.

16.5. Let (G, S) be a Coxeter system, C_0 is the initial chamber, and $v \in \overline{C}_0$. Show that the stabilizer of v in G is generated by *simple* reflections s_{α_i} such that $v \in \alpha_i^\perp$.

Solution: According to HW 12.1, $\mathrm{Stab}_G(v)$ is generated by reflections whose mirrors contain v. Let F be the face of C_0 of minimal dimension containing v, so we can write $F = \cap \alpha_i^\perp$, $i \in I$, where I is the corresponding index set. Let Γ_0 be the subgroup of G generated by r_{α_i}, $i \in I$. We claim that every reflection $r \in \mathrm{Stab}_G(v)$ is contained in Γ_0. Observe that since v belongs to the interior of F, the mirror of r contains the whole F.

Denote the dimension of F by k. The group Γ_0 is a reducible reflection group of rank $n-k$. Take a small neighborhood U of v such that $U \cap C_0$ is bounded by the mirrors of r_{α_i} only, $i \in I$. The copies of $U \cap C_0$ under action of Γ_0 tessellate U. If $r \notin \Gamma_0$, then adding r to Γ_0 we obtain a larger group Γ (fixing F). Therefore, $U \cap C_0$ must be cut by mirrors of reflections of Γ into smaller pieces. However, this implies that C_0 is cut by some mirror, which leads to a contradiction.

Alternatively, one could avoid any geometry. We assume below that G is a Weyl group, but exactly the same reasonings work for other finite Coxeter groups.

Let $g \in \mathrm{Stab}_G(v)$, we want to show that g is a product of simple reflections fixing v. We proceed by induction on $l(g) = n(g)$ (see Exercise 16.1). If $n(g) = 0$ then there is nothing to prove. If $n(g) > 0$, then there exists s_i such that $g\alpha_i \in \Delta^-$ (one set of simple roots cannot be positive with respect to another – check this!). By Exercise 16.1, $n(gs_i) = n(g) - 1$, and thus, by induction assumption, gs_i is a product of simple reflections fixing v. Since $v \in \overline{C}_0$, we have $(v, g\alpha_i) \le 0$. However, $(v, g\alpha_i) = (g^{-1}v, \alpha_i) = (v, \alpha_i) \ge 0$. Therefore, $(v, \alpha_i) = 0$, and thus $s_i \in \mathrm{Stab}_g(v)$, so $g = (gs_i)s_i$ is also a product of simple reflections fixing v.