# Topics in Combinatorics IV, Solutions 16 (Week 16) 

16.1. Let $\Delta$ be a root system, $\Pi$ is a set of simple roots, $\alpha_{i} \in \Pi$.
(a) ( $\star$ ) Show that $r_{\alpha_{i}}\left(\Delta^{+} \backslash \alpha_{i}\right)=\Delta^{+} \backslash \alpha_{i}$. In other words, $r_{\alpha_{i}}$ takes all positive roots except $\alpha$ to positive roots.
Hint: use Theorem 9.12.
(b) Let $w \in W, \alpha \in \Pi$. Denote $n(w)=\#\left\{\beta \in \Delta^{+} \mid w \beta \in \Delta^{-}\right\}$, i.e. the number of positive roots taken by $w$ to negative ones. Show that if $w \alpha \in \Delta^{+}$then $n\left(w r_{\alpha}\right)=n(w)+1$, and if $w \alpha \in \Delta^{-}$then $n\left(w r_{\alpha}\right)=n(w)-1$. In particular, $n(w) \leq l(w)$.
(c) Let $s_{1} \ldots s_{k}$ be a reduced expression for $w$, where $s_{i}=r_{\alpha_{i}}$ are simple reflections. Show that if $n(w)<l(w)$ then there exist $i<j$ such that $s_{i}\left(s_{i+1} \ldots s_{j-1}\right) \alpha_{j}=\alpha_{i}$.
(d) Show that $n(w)=l(w)$ for every $w \in W$.

## Solution:

(a) Let $\beta=\sum c_{j} \alpha_{j} \in \Delta^{+}$, i.e. $c_{j} \geq 0$. Since $\beta \neq \alpha_{i}$, there is $j \neq i$ such that $c_{j}>0$. Then $r_{\alpha_{i}}(\beta)=\beta-c \alpha_{i}$ has a positive coefficient $c_{j}$ before $\alpha_{j}$ (as it has not changed), and thus must be positive.
(b) Let $w \alpha \in \Delta^{+}$, and assume $w r_{\alpha}\left(\alpha^{\prime}\right) \in \Delta^{-}$for some $\alpha^{\prime} \in \Delta^{+}$. By (a), either $\alpha^{\prime}=\alpha$ (and then $w r_{\alpha}\left(\alpha^{\prime}\right)=w r_{\alpha}(\alpha)=w(-\alpha)=-w \alpha \in \Delta^{-}$), or $\alpha^{\prime \prime}:=r_{\alpha}\left(\alpha^{\prime}\right) \in \Delta^{+}$, and thus $\alpha^{\prime}=r_{a}\left(\alpha^{\prime \prime}\right)$, where $\alpha^{\prime \prime} \in \Delta^{+}$is taken by $w$ to $\Delta^{-}$. Therefore, $n\left(w r_{\alpha}\right)=n(w)+1$.
If $w \alpha \in \Delta^{-}$, then $w r_{\alpha} \alpha \in \Delta^{+}$, and, by the reasoning above, $n(w)=n\left(\left(w r_{\alpha}\right) r_{\alpha}\right)=n\left(w r_{\alpha}\right)+1$. As $n(w s) \leq n(w)+1$ for any simple reflection $s$, we see that $n(w) \leq l(w)$.
(c) Suppose that $n(w)<l(w)$. Applying (b), we see that there is some index $j$ such that $n\left(\left(s_{1} \ldots s_{j-1}\right) s_{j}\right) \neq n\left(s_{1} \ldots s_{j-1}\right)+1$, which implies that $\left(s_{1} \ldots s_{j-1}\right) \alpha_{j} \in \Delta^{-}$. Now, find the maximal $i$ such that $\left(s_{i+1} \ldots s_{j-1}\right) \alpha_{j} \in \Delta^{+}$and $s_{i}\left(s_{i+1} \ldots s_{j-1}\right) \alpha_{j} \in \Delta^{-}$(such $i$ exists since $\left.\alpha_{j}>0\right)$. By (a), $s_{i+1} \ldots s_{j-1} \alpha_{j}=\alpha_{i}$.
(d) By (b), $n(w) \leq l(w)$. If $n(w)<l(w)$ then, by $(c), s_{i+1} \ldots s_{j-1} \alpha_{j}=\alpha_{i}$, which implies $\left(s_{i+1} \ldots s_{j-1}\right) s_{j}\left(s_{j-1} \ldots s_{i+1}\right)=s_{i}$. Therefore, $s_{i} s_{i+1} \ldots s_{j-1} s_{j}=s_{i+1} \ldots s_{j_{1}}$, which means that the word was not reduced, so we get a contradiction.
16.2. Let $\Delta$ be a root system. Show that the highest root $\tilde{\alpha}_{0}$ is always long, i.e. $\left(\tilde{\alpha}_{0}, \tilde{\alpha}_{0}\right) \geq(\alpha, \alpha)$ for any $\alpha \in \Delta$.

## Solution:

Let $C_{0}$ be the Weyl chamber. Take any $\alpha \in \Delta$. Since $C_{0}$ is a fundamental domain for the action of the Weyl group $W$, there exists $w \in W$ such that $w(\alpha)=\beta \in \bar{C}_{0}$. Thus, we can assume $\left(\beta, \alpha_{i}\right) \geq 0$ for any $i$.
Now, since $\tilde{\alpha}_{0}$ is the highest root, we have $\tilde{\alpha}_{0}-\beta=\sum c_{i} \alpha_{i}$ with all $c_{i} \geq 0$. This implies $\left(\tilde{\alpha}_{0}-\beta, \beta\right)=$ $\left(\sum c_{i} \alpha_{i}, \beta\right)=\sum c_{i}\left(\alpha_{i}, \beta\right) \geq 0$, and $\left(\tilde{\alpha}_{0}-\beta, \tilde{\alpha}_{0}\right)=\left(\sum c_{i} \alpha_{i}, \tilde{\alpha}_{0}\right)=\sum c_{i}\left(\alpha_{i}, \tilde{\alpha}_{0}\right) \geq 0$. Therefore,

$$
\left(\tilde{\alpha}_{0}, \tilde{\alpha}_{0}\right) \geq\left(\beta, \tilde{\alpha}_{0}\right) \geq(\beta, \beta)=(\alpha, \alpha)
$$

16.3. Let $(G, S)$ be a Coxeter system, let $T \subset S$, and let $G_{T}$ be a standard parabolic subgroup (see HW 15.2). Define $G^{T}=\{g \in G \mid l(g t)>l(g) \forall t \in T\}$. Let $g \in G$.
(a) Let $u_{0} \in g G_{T}$ be a coset representative of minimal possible length across the whole coset. Show that $u_{0} \in G^{T}$ and $g=u_{0} v_{0}$ for some $v_{0} \in G_{T}$.
(b) Show that $l(g)=l\left(u_{0}\right)+l\left(v_{0}\right)$.
(c) Show that every $p \in g G_{T}$ can be written as $p=u_{0} v$ for some $v \in G_{T}$ with $l(p)=$ $l\left(u_{0}\right)+l(v)$.
(d) Show that $u_{0}$ is the unique element of $g G_{T}$ of minimal length.
(e) Show that there is a unique $u \in G^{T}$ and a unique $v \in G_{T}$ such that $g=u v$.

## Solution:

(a) Since $u_{0} \in g G_{T}$, we have $u_{0}=g v$ for $v \in G_{T}$, and thus $u_{0} v_{0}=g$ for $v_{0}=v^{-1} \in G_{T}$. Further, for $t \in T, u_{0} t=g v t=g(v t) \in g G_{T}$, so due to the minimality of length of $u_{0}$ we have $l\left(u_{0} t\right)>l\left(u_{0}\right)$, and thus $u_{0} \in G^{T}$.
(b) Let $s_{1} \ldots s_{k}$ and $t_{1} \ldots t_{m}$ be reduced expressions for $u_{0}$ and $v_{0}$ respectively. If $l(g)<l\left(u_{0}\right)+$ $l\left(v_{0}\right)$, then the word $s_{1} \ldots s_{k} t_{1} \ldots t_{m}$ is not reduced, so, by the Deletion Condition, we can remove two letters. If at least one of them is $s_{i}$, then we get $s_{1} \ldots \hat{s}_{i} \ldots s_{k} \sim_{G} g v^{\prime}$ for $v^{\prime} \in G_{T}$ in contradiction with the minimality of $u_{0}$. Therefore, $g \sim_{G} s_{1} \ldots s_{k} t_{1} \ldots \hat{t}_{i} \ldots \hat{t}_{j} \ldots t_{m}$, so $t_{1} \ldots \hat{t}_{i} \ldots \hat{t}_{j} \ldots t_{m} \sim_{G} u_{0}^{-1} g=v_{0}$, which contradicts the fact $t_{1} \ldots t_{m}$ is a reduced expression for $v_{0}$.
(c) This follows from the two steps above: we have only used that $g \in g G_{T}$.
(d) Due to (c), for every $u_{0}^{\prime} \in g G_{T}$ we have $u_{0}^{\prime}=u_{0} v$ with $l\left(u_{0}^{\prime}\right)=l\left(u_{0}\right)+l(v)$. If $u_{0}^{\prime}$ is minimal, then $l(v)=0$ and thus $u_{0}^{\prime}=u_{0}$.
(e) Suppose there is $u \in G^{T}$ and $v \in G_{T}$ such that $g=u v, u \neq u_{0}$. Then $u=g v^{-1} \in g G_{T}$. By (c), we can write $u=u_{0} v^{\prime}$, let $v^{\prime} \sim_{G} t_{1} \ldots t_{m}$ be a reduced expression. Then $l\left(u t_{m}\right) \leq$ $l\left(u_{0}\right)+(m-1)<l(u)$, so we come to a contradiction.
16.4. ( $\star$ ) Let $G$ be a finite Coxeter group, $(G, S)$ is a Coxeter system.
(a) Show that there is a unique element $g_{0}$ of maximal length. What is its length?
(b) Write down $g_{0}$ for the group of type $A_{3}$.

## Solution:

(a) Length of an element $g \in G$ is the number of reflections in its $R$-sequence appearing one time only, or, equivalently, the number of mirrors of reflections separating the initial chamber $C_{0}$ from $g C_{0}$. There is only one chamber separated from $C_{0}$ by mirrors of all reflections, this is $-C_{0}$. Therefore, $g_{0} C_{0}=-C_{0}$. In particular, $l\left(g_{0}\right)$ is the number of reflections in $G$.
(b) There are 6 reflections in $A_{3}$ (e.g., $S_{4}$ has six transpositions, or this can be seen by realizing $A_{3}$ as the symmetry group of a regular tetrahedron): these are

$$
s_{1}, s_{2}, s_{3}, s_{1} s_{2} s_{1}, s_{2} s_{3} s_{2}, s_{1} s_{2} s_{1} s_{3} s_{1} s_{2} s_{1}\left(=s_{1} s_{2} s_{3} s_{2} s_{1}\right)
$$

We can arrange them as follows:

$$
\begin{aligned}
& s_{1},\left(s_{1}\right) s_{2}\left(s_{1}\right),\left(s_{1} s_{2}\right) s_{3}\left(s_{2} s_{1}\right),\left(s_{1} s_{2} s_{3}\right) s_{2}\left(s_{3} s_{2} s_{1}\right)=s_{1} s_{3} s_{1}=s_{3} \\
& \left(s_{1} s_{2} s_{3} s_{2}\right) s_{1}\left(s_{2} s_{3} s_{2} s_{1}\right)=s_{1} s_{3} s_{2} s_{3} s_{1} s_{3} s_{2} s_{3} s_{1}=s_{1} s_{3} s_{2} s_{1} s_{2} s_{3} s_{1}=s_{1} s_{3} s_{1} s_{2} s_{1} s_{3} s_{1}=s_{3} s_{2} s_{3} \\
& \qquad\left(s_{1} s_{2} s_{3} s_{2} s_{1}\right) s_{2}\left(s_{1} s_{2} s_{3} s_{2} s_{1}\right)=s_{1} s_{2} s_{3} s_{1} s_{3} s_{2} s_{1}=s_{1} s_{2} s_{1} s_{2} s_{1}=s_{2}
\end{aligned}
$$

so $g_{0}=s_{1} s_{2} s_{3} s_{2} s_{1} s_{2}$. Of course, there are many ways to write down $g_{0}$ leading to distinct $R$-sequences.

Alternatively, one can construct directly the element taking $C_{0}$ to $-C_{0}$. We know that $\bar{W}\left(A_{3}\right)=S_{4}$ acts on $\mathbb{R}^{4} \cap H$ by permutations of coordinate axes, and that $C_{0}=\{x \in$ $\left.\mathbb{R}^{4} \mid x_{1}>x_{2}>x_{3}>x_{4}\right\} \cap H$, where $H=\left\{x \in \mathbb{R}^{4} \mid x_{1}+x_{2}+x_{3}+x_{4}=0\right\}$. Then $-C_{0}$ will be given by inequalities $x_{1}<x_{2}<x_{3}<x_{4}$, and thus $g_{0}=\binom{12344}{4321} \in S_{4}$. We are left to write $g_{0}$ down as a product of transpositions $s_{i}=(i i+1)$. Observe that $g_{0}=(14)(23)$, so we just need to write down (14) as a product of simple reflections. Clearly, (14) $=(13)(34)(13)=$ $((12)(23)(12))(34)((12)(23)(12))=(12)(23)(34)(23)(12)$, so $g_{0}=s_{1} s_{2} s_{3} s_{2} s_{1} s_{2}$ works indeed.
16.5. Let $(G, S)$ be a Coxeter system, $C_{0}$ is the initial chamber, and $v \in \bar{C}_{0}$. Show that the stabilizer of $v$ in $G$ is generated by simple reflections $s_{\alpha_{i}}$ such that $v \in \alpha_{i}^{\perp}$.

Solution: According to HW 12.1, $\operatorname{Stab}_{G}(v)$ is generated by reflections whose mirrors contain $v$. Let $F$ be the face of $C_{0}$ of minimal dimension containing $v$, so we can write $F=\cap \alpha_{i}^{\perp}, i \in I$, where $I$ is the corresponding index set. Let $\Gamma_{0}$ be the subgroup of $G$ generated by $r_{\alpha_{i}}, i \in I$. We claim that every reflection $r \in \operatorname{Stab}_{G}(v)$ is contained in $\Gamma_{0}$. Observe that since $v$ belongs to the interior of $F$, the mirror of $r$ contains the whole $F$.
Denote the dimension of $F$ by $k$. The group $\Gamma_{0}$ is a reducible reflection group of rank $n-k$. Take a small neighborhood $U$ of $v$ such that $U \cap C_{0}$ is bounded by the mirrors of $r_{\alpha_{i}}$ only, $i \in I$. The copies of $U \cap C_{0}$ under action of $\Gamma_{0}$ tessellate $U$. If $r \notin \Gamma_{0}$, then adding $r$ to $\Gamma_{0}$ we obtain a larger group $\Gamma$ (fixing $F$ ). Therefore, $U \cap C_{0}$ must be cut by mirrors of reflections of $\Gamma$ into smaller pieces. However, this implies that $C_{0}$ is cut by some mirror, which leads to a contradiction.

Alternatively, one could avoid any geometry. We assume below that $G$ is a Weyl group, but exactly the same reasonings work for other finite Coxeter groups.
Let $g \in \operatorname{Stab}_{G}(v)$, we want to show that $g$ is a product of simple reflections fixing $v$. We proceed by induction on $l(g)=n(g)$ (see Exercise 16.1). If $n(g)=0$ then there is nothing to prove. If $n(g)>0$, then there exists $s_{i}$ such that $g \alpha_{i} \in \Delta^{-}$(one set of simple roots cannot be positive with respect to another - check this!). By Exercise 16.1, $n\left(g s_{i}\right)=n(g)-1$, and thus, by induction assumption, $g s_{i}$ is a product of simple reflections fixing $v$. Since $v \in \bar{C}_{0}$, we have $\left(v, g \alpha_{i}\right) \leq 0$. However, $\left(v, g \alpha_{i}\right)=\left(g^{-1} v, \alpha_{i}\right)=\left(v, \alpha_{i}\right) \geq 0$. Therefore, $\left(v, \alpha_{i}\right)=0$, and thus $s_{i} \in \operatorname{Stab}_{g}(v)$, so $g=\left(g s_{i}\right) s_{i}$ is also a product of simple reflections fixing $v$.

