Durham University
Pavel Tumarkin Epiphany 2025

Topics in Combinatorics IV, Homework 16 (Week 16)

Due date for starred problems: **Monday, March 03, 12pm**.

16.1. Let Δ be a root system, Π is a set of simple roots, $\alpha_i \in \Pi$.

(a) ($\star$) Show that $r_{\alpha_i}(\Delta^+ \setminus \alpha_i) = \Delta^+ \setminus \alpha_i$. In other words, r_{α_i} takes all positive roots except α to positive roots.
Hint: use Theorem 9.12.

(b) Let $w \in W$, $\alpha \in \Pi$. Denote $n(w) = \#\{\beta \in \Delta^+ \mid w\beta \in \Delta^-\}$, i.e. the number of positive roots taken by w to negative ones. Show that if $w\alpha \in \Delta^+$ then $n(wr_\alpha) = n(w)+1$, and if $w\alpha \in \Delta^-$ then $n(wr_\alpha) = n(w) - 1$. In particular, $n(w) \le l(w)$.

(c) Let $s_1 \dots s_k$ be a reduced expression for w, where $s_i = r_{\alpha_i}$ are simple reflections. Show that if $n(w) < l(w)$ then there exist $i < j$ such that $s_i(s_{i+1} \dots s_{j-1})\alpha_j = \alpha_i$.

(d) Show that $n(w) = l(w)$ for every $w \in W$.

16.2. Let Δ be a root system. Show that the highest root $\tilde{\alpha}_0$ is always long, i.e. $(\tilde{\alpha}_0, \tilde{\alpha}_0) \ge (\alpha, \alpha)$ for any $\alpha \in \Delta$.

16.3. Let (G, S) be a Coxeter system, let $T \subset S$, and let G_T be a standard parabolic subgroup (see HW 15.2). Define $G^T = \{g \in G \mid l(gt) > l(g) \forall t \in T\}$. Let $g \in G$.

(a) Let $u_0 \in gG_T$ be a coset representative of minimal possible length across the whole coset. Show that $u_0 \in G^T$ and $g = u_0 v_0$ for some $v_0 \in G_T$.

(b) Show that $l(g) = l(u_0) + l(v_0)$.

(c) Show that every $p \in gG_T$ can be written as $p = u_0 v$ for some $v \in G_T$ with $l(p) = l(u_0) + l(v)$.

(d) Show that u_0 is the unique element of gG_T of minimal length.

(e) Show that there is a unique $u \in G^T$ and a unique $v \in G_T$ such that $g = uv$.

16.4. ($\star$) Let G be a finite Coxeter group, (G, S) is a Coxeter system.

(a) Show that there is a unique element g_0 of maximal length. What is its length?

(b) Write down g_0 for the group of type A_3.

16.5. Let (G, S) be a Coxeter system, C_0 is the initial chamber, and $v \in \overline{C}_0$. Show that the stabilizer of v in G is generated by *simple* reflections s_{α_i} such that $v \in \alpha_i^\perp$.