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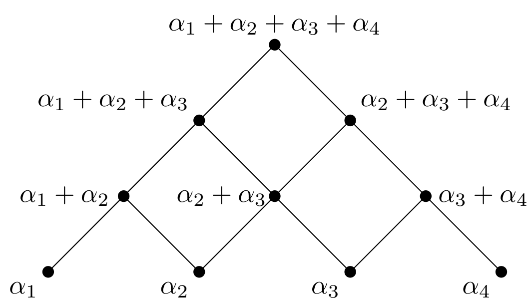
Topics in Combinatorics IV, Solutions 17 (Week 17)

17.1. ($\star$) Draw the Hasse diagram of the root poset of root system A_4.

Solution: Every root $e_i - e_j$ for $i < j$ can be written as

$$e_i - e_j = \sum_{k=i}^{j-1} e_k - e_{k+1} = \sum_{k=i}^{j-1} \alpha_k$$

which implies that $e_i - e_j \geq e_k - e_l$ if and only if $i \leq k$ and $j \geq l$. We get the following Hasse diagram:



17.2. ($\star$) Let (W,S) be a Coxeter system. A subgroup H of W is a *parabolic subgroup* if it is conjugated to a standard parabolic subgroup W_T for some $T \subset S$ (see HW 15.2), i.e. $H = w^{-1}W_T w$ for some $w \in W$. Show that for any $p \in \mathbb{R}^n$ the stabilizer $\mathrm{Stab}_W(p)$ is a parabolic subgroup.

Solution: This follows from HW 16.5. Let C be the initial chamber of W. There exists $w \in W$ such that $wp \in \overline{C}$. By HW 16.5, the stabilizer of wp is a standard parabolic subgroup W_T, where $s \in T$ if and only if $s(wp) = wp$. Therefore, the stabilizer of p is precisely $w^{-1}W_T w$.

17.3. Let (W,S) be an irreducible Coxeter system. Denote $c_n = \#\{w \in W \mid l(w) = n\}$, and define the generating function

$$W(q) = \sum_{n \geq 0} c_n q^n = \sum_{w \in W} q^{l(w)},$$

which is called the *Poincaré series* of W. In the case when W is finite, $W(q)$ is called the *Poincaré polynomial* of W.

Recall that if $T \subset S$ then W_T denotes a standard parabolic subgroup, and $W^T = \{w \in W \mid l(wt) > l(w) \forall t \in T\}$ (see HW 16.2).

For every $X \subset W$ denote also $X(q) = \sum_{w \in X} q^{l(w)}$.

(a) Show that if $T \subset S$ then $W(q) = W_T(q)W^T(q)$.

(b) Let $w \in W$, define $F = F(w) = \{s \in S \mid l(ws) > l(w)\}$. Show that $\sum_{T \subset F} (-1)^{|T|} = 0$ unless W is finite and $w = w_0$ is the longest element of W.

(c) Show that

$$\sum_{T \subset S} (-1)^{|T|} \frac{W(q)}{W_T(q)} = \sum_{T \subset S} (-1)^{|T|} W^T(q) = \begin{cases} 0 & \text{if } W \text{ is infinite,} \\ q^N & \text{if } W \text{ is finite,} \end{cases}$$

where N is the length of the longest element of W.

(d) Assume W is finite. Show that

$$\sum_{T \subset S} (-1)^{|T|} \frac{|W|}{|W_T|} = 1$$

(e) Apply the formula from (d) to compute the order of the group H_3. Can you compute the order of H_4 in this way?

Solution:

(a) This follows from HW 16.2: take any $w \in W$, then there are unique $u \in W_T$ and $v \in W^T$ such that $w = uv$. Since $l(w) = l(u) + l(v)$, we have $q^{l(w)} = q^{l(u)}q^{l(v)}$, so

$$W(q) = \sum_{w \in W} q^{l(w)} = \sum_{u \in W_T, v \in W^T} q^{l(u)}q^{l(v)} = \sum_{u \in W_T} q^{l(u)} \sum_{v \in W^T} q^{l(v)} = W_T(q)W^T(q)$$

(b) Suppose F is empty, then $l(ws) < l(w)$ for any $s \in S$, and thus W is finite and $w = w_0$. Otherwise, F is non-empty, and the statement claims that the alternating sum of binomial coefficients is zero, which follows from the identity $(1-1)^{|F|} = 0$.

(c) The first equality follows from (a). To prove the second equality, write

$$\sum_{T \subset S} (-1)^{|T|} W^T(q) = \sum_{T \subset S} (-1)^{|T|} \left(\sum_{w \in W^T} q^{l(w)} \right)$$

and observe that $w \in W^T$ if and only if $T \subset F(w)$. Therefore, for a given w the coefficient of $q^{l(w)}$ is precisely $\sum_{T \subset F(w)} (-1)^{|T|}$, which vanishes for every $w \in W$ unless W is finite due to (b). If W is finite, then the only $w \in W$ such that $F(w)$ is empty is w_0, and thus the only non-zero term is $q^{l(w_0)}$.

(d) Plug in $q = 1$ in (c).

(e) Let $H_3 = \langle s_1, s_2, s_3 \mid s_i^2, (s_1s_2)^3, (s_2s_3)^5, (s_1s_3)^2 \rangle$. Then we have the following subsets of S and the orders of the standard parabolic subgroups:

T	$\emptyset$	$\{s_1\}$	$\{s_2\}$	$\{s_3\}$	$\{s_1, s_2\}$	$\{s_1, s_3\}$	$\{s_2, s_3\}$	$\{s_1, s_2, s_3\}$
W_T	$\{e\}$	A_1	A_1	A_1	A_2	$A_1 \times A_1$	$I_2(5)$	H_3
$\lvert W_T \rvert$	1	2	2	2	6	4	10	$\lvert W(H_3) \rvert$

Therefore, we have

$$|W|\left(\frac{1}{1}-\frac{1}{2}-\frac{1}{2}-\frac{1}{2}+\frac{1}{6}+\frac{1}{4}+\frac{1}{10}\right)-1=1,$$

which is equivalent to $|W(H_3)|\cdot\frac{2}{120}=2$, so we get $|W(H_3)|=120$.

The group $W(H_4)$ has rank 4, which implies that the summand $\frac{|W|}{|W|}$ in the sum will have coefficient $+1$. Therefore, the formula in (c) will become $|W|\cdot(\dots)=0$, which cannot be used for computation of the order.

17.4. Let Δ be a root system. Let $(\cdot,\cdot)$ be the dot product, and let $\langle\alpha\mid\beta\rangle=\dfrac{2(\alpha,\beta)}{(\beta,\beta)}$ for $\alpha,\beta\in\Delta$.

(a) Let $\alpha,\beta\in\Delta$ be non-collinear. Show that if $(\alpha,\beta)<0$ then $\alpha+\beta\in\Delta$, and if $(\alpha,\beta)>0$ then $\alpha-\beta\in\Delta$.

(b) Show that there exist integers $p,q\geq 0$, such that the set $I=\{k\in\mathbb{Z}\mid\beta+k\alpha\in\Delta\}$ is an interval $[-q,p]\cap\mathbb{Z}$.

(c) Let $R=\{\beta+k\alpha\mid k\in I\}$. Show that $r_\alpha(R)=R$. Show that $q-p=\langle\beta\mid\alpha\rangle$.

Solution.

(a) If $(\alpha,\beta)<0$, then either $\langle\alpha\mid\beta\rangle=-1$ or $\langle\beta\mid\alpha\rangle=-1$. In the former case $r_\beta(\alpha)=\alpha-\langle\alpha\mid\beta\rangle\beta=\alpha+\beta$, while in the latter case $r_\alpha(\beta)=\beta-\langle\beta\mid\alpha\rangle\alpha=\beta+\alpha$.

Similarly, if $(\alpha,\beta)>0$, then either $\langle\alpha\mid\beta\rangle=1$ or $\langle\beta\mid\alpha\rangle=1$. In the former case $r_\beta(\alpha)=\alpha-\langle\alpha\mid\beta\rangle\beta=\alpha-\beta$, while in the latter case $r_\alpha(\beta)=\beta-\langle\beta\mid\alpha\rangle\alpha=\beta-\alpha$ (and thus $-(\beta-\alpha)=\alpha-\beta\in\Delta$ as well).

(b) Denote by p and $-q$ the maximal and minimal elements of I respectively (note that $0\in I$ as $\beta\in\Delta$). Assume that there are some integers in the interval $[-q,p]$ not belonging to I. Let the gap contain a maximal interval $[a+1,b-1]$, where $a,b\in I$, $b>a+1$. By (a), this implies $(\beta+a\alpha,\alpha)\geq 0$ and $(\beta+b\alpha,\alpha)\leq 0$. However,

$$(\beta+a\alpha,\alpha)-(\beta+b\alpha,\alpha)=((a-b)\alpha,\alpha)=(a-b)(\alpha,\alpha)<0,$$

so we get a contradiction.

(c) By definition, $r_\alpha(\beta+k\alpha)=\beta-\langle\beta\mid\alpha\rangle\alpha-k\alpha=\beta+k'\alpha$, where $k'=-k-\langle\beta\mid\alpha\rangle$. Thus, r_α maps R to itself, and the map $k\mapsto k'$ is decreasing on I, so $r_\alpha(\beta+p\alpha)=\beta-q\alpha$. Therefore, $-q=-p-\langle\beta\mid\alpha\rangle$, so $q-p=\langle\beta\mid\alpha\rangle$ as required.