Durham University
Pavel Tumarkin Epiphany 2025

Topics in Combinatorics IV, Homework 17 (Week 17)

Due date for starred problems: **Monday, March 17, 12pm**.

17.1. ($\star$) Draw the Hasse diagram of the root poset of root system A_4.

17.2. ($\star$) Let (W,S) be a Coxeter system. A subgroup H of W is a *parabolic subgroup* if it is conjugated to a standard parabolic subgroup W_T for some $T \subset S$ (see HW 15.2), i.e. $H = w^{-1}W_T w$ for some $w \in W$. Show that for any $p \in \mathbb{R}^n$ the stabilizer $\mathrm{Stab}_W(p)$ is a parabolic subgroup.

17.3. Let (W,S) be an irreducible Coxeter system. Denote $c_n = \#\{w \in W \mid l(w) = n\}$, and define the generating function

$$W(q) = \sum_{n \geq 0} c_n q^n = \sum_{w \in W} q^{l(w)},$$

which is called the *Poincaré series* of W. In the case when W is finite, $W(q)$ is called the *Poincaré polynomial* of W.

Recall that if $T \subset S$ then W_T denotes a standard parabolic subgroup, and $W^T = \{w \in W \mid l(wt) > l(w) \forall t \in T\}$ (see HW 16.2).

For every $X \subset W$ denote also $X(q) = \sum\limits_{w \in X} q^{l(w)}$.

(a) Show that if $T \subset S$ then $W(q) = W_T(q)W^T(q)$.

(b) Let $w \in W$, define $F = F(w) = \{s \in S \mid l(ws) > l(w)\}$. Show that $\sum\limits_{T \subset F} (-1)^{|T|} = 0$ unless W is finite and $w = w_0$ is the longest element of W.

(c) Show that

$$\sum_{T \subset S} (-1)^{|T|} \frac{W(q)}{W_T(q)} = \sum_{T \subset S} (-1)^{|T|} W^T(q) = \begin{cases} 0 & \text{if } W \text{ is infinite,} \\ q^N & \text{if } W \text{ is finite,} \end{cases}$$

where N is the length of the longest element of W.

(d) Assume W is finite. Show that

$$\sum_{T \subset S} (-1)^{|T|} \frac{|W|}{|W_T|} = 1$$

(e) Apply the formula from (d) to compute the order of the group H_3. Can you compute the order of H_4 in this way?

17.4. Let Δ be a root system. Let $(\cdot, \cdot)$ be the dot product, and let $\langle \alpha \mid \beta \rangle = \dfrac{2(\alpha, \beta)}{(\beta, \beta)}$ for $\alpha, \beta \in \Delta$.

(a) Let $\alpha, \beta \in \Delta$ be non-collinear. Show that if $(\alpha, \beta) < 0$ then $\alpha + \beta \in \Delta$, and if $(\alpha, \beta) > 0$ then $\alpha - \beta \in \Delta$.

(b) Show that there exist integers $p, q \geq 0$, such that the set $I = \{k \in \mathbb{Z} \mid \beta + k\alpha \in \Delta\}$ is an interval $[-q, p] \cap \mathbb{Z}$.

(c) Let $R = \{\beta + k\alpha \mid k \in I\}$. Show that $r_\alpha(R) = R$. Show that $q - p = \langle \beta \mid \alpha \rangle$.