## Topics in Combinatorics IV, Solutions 18 (Week 18)

18.1. ( $\star$ ) Let $\Delta$ be a root system, $\Pi=\left\{\alpha_{i}\right\}$ are simple roots.
(a) Let $\alpha, \beta \in \Delta^{+}, \alpha<\cdot \beta$ in the root poset, $\beta-\alpha=\sum_{j \in J} c_{j} \alpha_{j}$ for some $J \subset[n]$, where $c_{j}>0$ for all $j \in J$. Show that either $\beta-\alpha=\alpha_{j}$ (i.e. $|J|=1$ and $c_{j}=1$ ), or $\left(\beta-\alpha, \alpha_{j}\right) \leq 0$ for any $j \in J$.
Hint: Use Exercise 9.15 from lectures.
(b) Show that the root poset of a root system is ranked, where the rank of $\alpha=\sum c_{i} \alpha_{i}$ is $\sum c_{i}$ (this number is called the height of $\alpha$ and is denoted by ht $\alpha$ ).
Hint: Compute $(\beta-\alpha, \beta-\alpha)$ in (a).

## Solution:

(a) Suppose that $\beta-\alpha \neq \alpha_{j}$ for any $j \in J$. Since $\alpha<\cdot \beta$, for any $j \in J$ we have $\alpha+\alpha_{j} \notin \Delta$ and $\beta-\alpha_{j} \notin \Delta$, which, by Exercise 9.15, implies $\left(\alpha, \alpha_{j}\right) \geq 0$ and $\left(\beta, \alpha_{j}\right) \leq 0$. Therefore, $\left(\beta-\alpha, \alpha_{j}\right) \leq 0$.
(b) We need to prove that $\alpha<\cdot \beta$ implies ht $\beta=\mathrm{ht} \alpha+1$, which is equivalent to $\beta=\alpha+\alpha_{j}$ for some $\alpha_{j}$ (as $\alpha<\beta$ ).
Let $\beta-\alpha=\sum_{j \in J} c_{j} \alpha_{j}$ for some $J \subset[n]$. If we suppose that $\beta-\alpha \neq \alpha_{j}$ for any $j \in J$, then (a) implies that $\left(\beta-\alpha, \alpha_{j}\right) \leq 0$ for any $j \in J$. Following the hint, we compute

$$
(\beta-\alpha, \beta-\alpha)=\left(\beta-\alpha, \sum_{j \in J} c_{j} \alpha_{j}\right)=\sum_{j \in J} c_{j}\left(\beta-\alpha, \alpha_{j}\right) \leq 0
$$

which implies $\beta-\alpha=0$, so we came to a contradiction with $\alpha<\beta$.
18.2. Let $\Delta$ be a root system, $\Pi=\left\{\alpha_{i}\right\}$ are simple roots, $W$ is the Weyl group, and $\Sigma$ is the Dynkin diagram of $\Delta$.
(a) Let $I \subset[n]$ be some index set such that $\left(\alpha_{i}, \alpha_{j}\right)=0$ for all $i, j \in I$. Show that the standard parabolic subgroup $W_{I}$ is isomorphic to $\left(\mathbb{Z}_{2}\right)^{|I|}$.
Hint: use HW 15.2.
(b) Let $\alpha \in \Delta, \alpha=\sum c_{i} \alpha_{i}$. Define the support of $\alpha$ to be the set $I \subset[n]$ such that $i \in I$ if and only if $c_{i} \neq 0$. Show that if $\left(\alpha_{i}, \alpha_{j}\right)=0$ for all $i, j \in I$ then $\alpha= \pm \alpha_{i}$ for some $i \in[n]$.
(c) Let $\alpha \in \Delta, \alpha=\sum c_{i} \alpha_{i}, I$ is the support of $\alpha$. Show that vertices of $\Sigma$ corresponding to $\alpha_{i}, i \in I$, form a connected subgraph of $\Sigma$.

## Solution:

(a) By HW 15.2, $W_{I}$ is isomorphic to the Weyl group of the root system with Dynkin diagram spanned by vertices corresponding to $\alpha_{i}, i \in I$. This diagram consists of $|I|$ vertices without any edges, so it is a direct product of $|I|$ copies of $W\left(A_{1}\right) \simeq S_{2} \simeq \mathbb{Z}_{2}$.
(b) By (a), the only reflections in $W_{I}$ are $r_{\alpha_{i}}$.
(c) Suppose the support $I$ of $\alpha$ is disconnected, e.g. $I=I_{1} \cup I_{2}$ and $I_{1}$ is not connected to $I_{2}$ in $\Sigma$. By HW 15.2, $W_{I}=W_{I_{1}} \times W_{I_{2}}$, so all reflections in $W_{I}$ belong either to $W_{I_{1}}$ or to $W_{I_{2}}$. Therefore, either $I \subset I_{1}$ or $I \subset I_{2}$.
18.3. Let $(W, S)$ be an irreducible Coxeter system. The goal of this exerxise is to show that if there is a quadratic form $Q$ on $\mathbb{R}^{n}$ invariant under $W$, then $Q(x)=c(x, x)$ for some $c \in \mathbb{R}$. Given a quadratic form $Q$, we will abuse notation by writing $Q(x, y)$ instead of $\frac{1}{2}(Q(x+y)-Q(x)-Q(y))$.
(a) Let $Q$ be a quadratic form invariant under $W$, i.e. $Q(x, y)=Q(w x, w y)$ for every $w \in W$. Recall that $Q(x, y)=(A x, y)$ for some $A \in M_{n}(\mathbb{R})$. Show that $w(A x)=A(w x)$ for every $x \in \mathbb{R}^{n}, w \in W$.
(b) For $s_{i} \in S$ let $s_{i}=r_{\alpha_{i}},\left\|\alpha_{i}\right\|^{2}=2$. Show that $r_{\alpha_{i}}\left(A \alpha_{i}\right)=-A \alpha_{i}$, and $A \alpha_{i}=c_{i} \alpha_{i}$ for some $c_{i} \in \mathbb{R}$.
(c) Let $\left(s_{i} s_{j}\right)^{2} \neq e$. Show that $c_{i}=c_{j}$. Deduce from this that $Q(x)=c(x, x)$.

Hint: compute $A s_{i}\left(\alpha_{j}\right)$ and $s_{i}\left(A \alpha_{j}\right)$.

## Solution:

(a) Since $W$ is a subgroup of $O_{n}(\mathbb{R}), Q(x, y)=(A x, y)=(w(A x), w y)$. On the other hand, $Q(x, y)=Q(w x, w y)=(A(w x), w y)$. As $w y$ can be an arbitrary vector, we conclude that $w(A x)=A(w x)$.
(b) Due to (a), $r_{\alpha_{i}}\left(A \alpha_{i}\right)=A\left(r_{\alpha_{i}}\left(\alpha_{i}\right)\right)=A\left(-\alpha_{i}\right)=-\alpha_{i}$. A reflection in $\mathbb{R}^{n}$ has eigenvalue -1 of multiplicity one and eigenvalue 1 of multiplicity $n-1$. Therefore, $A \alpha_{i}$ is proportional to the $\alpha_{i}$ which is an eigenvector with eigenvalue -1 .
(c) We have $A s_{i}\left(\alpha_{j}\right)=A\left(\alpha_{j}-\left(\alpha_{i}, \alpha_{j}\right) \alpha_{i}\right)=c_{j} \alpha_{j}-\left(\alpha_{i}, \alpha_{j}\right) c_{i} \alpha_{i}$. Further, $s_{i}\left(A \alpha_{j}\right)=s_{i}\left(c_{j} \alpha_{j}\right)=$ $c_{j} s_{i}\left(\alpha_{j}\right)=c_{j} \alpha_{j}-c_{j}\left(\alpha_{i}, \alpha_{j}\right) \alpha_{i}$. By (a), the obtained expressions are equal. Since $\mathrm{j}\left(\alpha_{i}, \alpha_{j}\right) \neq 0$ by assumption, we get $c_{i}=c_{j}$. Finally, as $W$ is irreducible, the Coxeter diagram is connected, so for any $s_{i}$ and $s_{j}$ there is a path $s_{i}=s_{i_{1}}, s_{i_{2}}, \ldots, s_{i_{k}}=s_{j}$ such that $s_{i_{l}}$ and $s_{i_{l+1}}$ do not commute. Therefore, all $c_{i}=c$, so $A x=c x$, and thus $Q(x)=Q(x, x)=(A x, x)=(c x, x)=$ $c(x, x)$ for every $x \in \mathbb{R}^{n}$.

