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Topics in Combinatorics IV, Solutions 18 (Week 18)

18.1. ($\star$) Let Δ be a root system, $\Pi = \{\alpha_i\}$ are simple roots.

(a) Let $\alpha, \beta \in \Delta^+$, $\alpha \lessdot \beta$ in the root poset, $\beta - \alpha = \sum\limits_{j \in J} c_j \alpha_j$ for some $J \subset [n]$, where $c_j > 0$ for all $j \in J$. Show that either $\beta - \alpha = \alpha_j$ (i.e. $|J| = 1$ and $c_j = 1$), or $(\beta - \alpha, \alpha_j) \le 0$ for any $j \in J$.
Hint: Use Exercise 9.15 from lectures.

(b) Show that the root poset of a root system is ranked, where the rank of $\alpha = \sum c_i \alpha_i$ is $\sum c_i$ (this number is called the *height* of α and is denoted by $\operatorname{ht} \alpha$).
Hint: Compute $(\beta - \alpha, \beta - \alpha)$ in (a).

Solution:

(a) Suppose that $\beta - \alpha \neq \alpha_j$ for any $j \in J$. Since $\alpha \lessdot \beta$, for any $j \in J$ we have $\alpha + \alpha_j \notin \Delta$ and $\beta - \alpha_j \notin \Delta$, which, by Exercise 9.15, implies $(\alpha, \alpha_j) \ge 0$ and $(\beta, \alpha_j) \le 0$. Therefore, $(\beta - \alpha, \alpha_j) \le 0$.

(b) We need to prove that $\alpha \lessdot \beta$ implies $\operatorname{ht} \beta = \operatorname{ht} \alpha + 1$, which is equivalent to $\beta = \alpha + \alpha_j$ for some α_j (as $\alpha < \beta$).
Let $\beta - \alpha = \sum\limits_{j \in J} c_j \alpha_j$ for some $J \subset [n]$. If we suppose that $\beta - \alpha \neq \alpha_j$ for any $j \in J$, then (a) implies that $(\beta - \alpha, \alpha_j) \le 0$ for any $j \in J$. Following the hint, we compute

$$(\beta - \alpha, \beta - \alpha) = \left(\beta - \alpha, \sum_{j \in J} c_j \alpha_j\right) = \sum_{j \in J} c_j (\beta - \alpha, \alpha_j) \le 0,$$

which implies $\beta - \alpha = 0$, so we came to a contradiction with $\alpha \lessdot \beta$.

18.2. Let Δ be a root system, $\Pi = \{\alpha_i\}$ are simple roots, W is the Weyl group, and Σ is the Dynkin diagram of Δ.

(a) Let $I \subset [n]$ be some index set such that $(\alpha_i, \alpha_j) = 0$ for all $i, j \in I$. Show that the standard parabolic subgroup W_I is isomorphic to $(\mathbb{Z}_2)^{|I|}$.
Hint: use HW 15.2.

(b) Let $\alpha \in \Delta$, $\alpha = \sum c_i \alpha_i$. Define the *support* of α to be the set $I \subset [n]$ such that $i \in I$ if and only if $c_i \neq 0$. Show that if $(\alpha_i, \alpha_j) = 0$ for all $i, j \in I$ then $\alpha = \pm \alpha_i$ for some $i \in [n]$.

(c) Let $\alpha \in \Delta$, $\alpha = \sum c_i \alpha_i$, I is the support of α. Show that vertices of Σ corresponding to α_i, $i \in I$, form a connected subgraph of Σ.

Solution:

(a) By HW 15.2, W_I is isomorphic to the Weyl group of the root system with Dynkin diagram spanned by vertices corresponding to $\alpha_i, i \in I$. This diagram consists of $|I|$ vertices without any edges, so it is a direct product of $|I|$ copies of $W(A_1) \simeq S_2 \simeq \mathbb{Z}_2$.

(b) By (a), the only reflections in W_I are r_{α_i}.

(c) Suppose the support I of α is disconnected, e.g. $I = I_1 \cup I_2$ and I_1 is not connected to I_2 in Σ. By HW 15.2, $W_I = W_{I_1} \times W_{I_2}$, so all reflections in W_I belong either to W_{I_1} or to W_{I_2}. Therefore, either $I \subset I_1$ or $I \subset I_2$.

18.3. Let (W, S) be an irreducible Coxeter system. The goal of this exerxise is to show that if there is a quadratic form Q on $\mathbb{R}^n$ invariant under W, then $Q(x) = c(x, x)$ for some $c \in \mathbb{R}$. Given a quadratic form Q, we will abuse notation by writing $Q(x, y)$ instead of $\frac{1}{2}(Q(x + y) - Q(x) - Q(y))$.

(a) Let Q be a quadratic form invariant under W, i.e. $Q(x, y) = Q(wx, wy)$ for every $w \in W$. Recall that $Q(x, y) = (Ax, y)$ for some $A \in M_n(\mathbb{R})$. Show that $w(Ax) = A(wx)$ for every $x \in \mathbb{R}^n, w \in W$.

(b) For $s_i \in S$ let $s_i = r_{\alpha_i}$, $\|\alpha_i\|^2 = 2$. Show that $r_{\alpha_i}(A\alpha_i) = -A\alpha_i$, and $A\alpha_i = c_i\alpha_i$ for some $c_i \in \mathbb{R}$.

(c) Let $(s_i s_j)^2 \neq e$. Show that $c_i = c_j$. Deduce from this that $Q(x) = c(x, x)$.
Hint: compute $As_i(\alpha_j)$ and $s_i(A\alpha_j)$.

Solution:

(a) Since W is a subgroup of $O_n(\mathbb{R})$, $Q(x, y) = (Ax, y) = (w(Ax), wy)$. On the other hand, $Q(x, y) = Q(wx, wy) = (A(wx), wy)$. As wy can be an arbitrary vector, we conclude that $w(Ax) = A(wx)$.

(b) Due to (a), $r_{\alpha_i}(A\alpha_i) = A(r_{\alpha_i}(\alpha_i)) = A(-\alpha_i) = -\alpha_i$. A reflection in $\mathbb{R}^n$ has eigenvalue -1 of multiplicity one and eigenvalue 1 of multiplicity $n - 1$. Therefore, $A\alpha_i$ is proportional to the α_i which is an eigenvector with eigenvalue -1.

(c) We have $As_i(\alpha_j) = A(\alpha_j - (\alpha_i, \alpha_j)\alpha_i) = c_j\alpha_j - (\alpha_i, \alpha_j)c_i\alpha_i$. Further, $s_i(A\alpha_j) = s_i(c_j\alpha_j) = c_j s_i(\alpha_j) = c_j\alpha_j - c_j(\alpha_i, \alpha_j)\alpha_i$. By (a), the obtained expressions are equal. Since j$(\alpha_i, \alpha_j) \neq 0$ by assumption, we get $c_i = c_j$. Finally, as W is irreducible, the Coxeter diagram is connected, so for any s_i and s_j there is a path $s_i = s_{i_1}, s_{i_2}, \ldots, s_{i_k} = s_j$ such that s_{i_l} and $s_{i_{l+1}}$ do not commute. Therefore, all $c_i = c$, so $Ax = cx$, and thus $Q(x) = Q(x, x) = (Ax, x) = (cx, x) = c(x, x)$ for every $x \in \mathbb{R}^n$.