Durham University
Pavel Tumarkin

Epiphany 2025

Topics in Combinatorics IV, Solutions 19 (Week 19)

Throughout the problem sheet Δ is a root system of rank n, $\Pi = \{\alpha_i\}$ are simple roots, $\tilde{\alpha}_0$ is the highest root, W is the Weyl group, h is the Coxeter number.

19.1. Compute the Coxeter number and exponents of the Weyl group of type

(a) C_4;

(b) C_n.

Solution: Let $\{e_i\}$ be an orthnormal basis of $\mathbb{R}^n$, and let $s_i = r_{\alpha_i}$, where $\alpha_i = e_i - e_{i+1}$ for $i < n$ and $\alpha_n = 2e_n$. Take $c = s_1 \dots s_{n-1}s_n = (s_1 \dots s_{n-1})s_n$, where $s_1 \dots s_{n-1}$ is a cyclic permutation of coordinates $1, \dots, n$, and s_n is the change of sign of n-th coordinate. Therefore, for $n = 4$ we have

$$c = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The characteristic polynomial of c is λ^4+1, so the eigenvalues are $\exp(\frac{2\pi i}{8}+k\frac{2\pi i}{4})$, where $k = 0, 1, 2, 3$. Thus, the exponents are $1, 3, 5, 7$, and the Coxeter number is $h = \frac{2}{n}(m_1 + m_2 + m_3 + m_4) = \frac{1}{2}(1 + 3 + 5 + 7) = 8$.

For arbitrary n, the characteristic polynomial of c is $(-1)^n(\lambda^n + 1)$, so the eigenvalues of c are $\exp(\frac{2\pi i}{2n} + k\frac{2\pi i}{n})$, where $k = 0, \dots, n-1$, and the corresponding exponents are $2k + 1$. The Coxeter number is $h = \frac{2}{n}(m_1 + \dots + m_n) = \frac{2}{n}(1 + 3 + \dots + (2n-1)) = \frac{2}{n}(\frac{n}{2} \cdot 2n) = 2n$.

19.2. (a) Show that the Coxeter number of the Weyl group of type E_8 is equal to the Coxeter number of the Coxeter group of type H_4.

(b) Show that the symmetric group S_{n+1} contains a subgroup isomorphic to the dihedral group $I_2(n+1)$.

(c) Let $W = \langle s_1, \dots, s_4 \mid s_i^2, (s_2s_j)^3 \text{ for } j \neq 2, (s_ks_l)^2 \text{ for } k, l \neq 2\rangle$ be the Weyl group of type D_4. Show that the subgroup of W generated by s_1, s_2 and s_3s_4 is isomorphic to the Weyl group of type B_3.

Solution:

(a) By the construction of H_4 as a subgroup of E_8, the generators of H_4 are s_it_i (see Section 10.3.1 of lecture notes), so a Coxeter element of H_4 is $s_1t_1 \dots s_4t_4$. However, this is a Coxeter element of E_8 as well.

(b) S_{n+1} is a Weyl group of type A_n, its Coxeter number is $n + 1$. If we take a bipartite Coxeter element $c = c'c''$, then $c'^2 = c''^2 = c^{n+1} = e$, so c' and c'' generate a group Γ which is a quotient of $I_2(n + 1)$. There are no more relations on c' and c'': Γ contains $n + 1$ elements of type c^k, and also $c' \neq c^k$ for any k, so there are at least $2(n + 1)$ elements.

(c) Let $\{e_i\}$ be an orthonormal basis of $\mathbb{R}^4$, and let $s_i = r_{\alpha_i}$, where α_i are simple roots of D_4, i.e.

$$\alpha_1 = e_1 - e_2,\ \alpha_2 = e_2 - e_3,\ \alpha_3 = e_3 - e_4,\ \alpha_4 = e_3 + e_4.$$

Then s_1, s_2 act on $\mathbb{R}^4$ by transpositions of coordinates, and s_3s_4 changes the signs of e_3 and e_4 simultaneously.

Denote by Γ the subgroup of W generated by s_1, s_2, s_3s_4. Let $L \subset \mathbb{R}^4$ be a 3-dimentional subspace spanned by e_1, e_2, e_3, observe that the action of Γ on L is precisely the action of B_3: s_1, s_2 act on $\mathbb{R}^3$ by transpositions of coordinates, and s_3s_4 changes the sign of e_3. We are left to check that any element acting on L trivially acts on e_4 trivially as well. Indeed, all relations in B_3 involving r_{e_3} contain even number of it, so if an element of Γ acts trivially on L then it involves an even number of s_3s_4, which implies that the sign of e_4 will be changed an even number of times, thus leaving e_4 intact.

19.3. (a) Define $\gamma = \sum\limits_{\beta\in\Delta^+} \frac{\beta}{(\beta,\beta)}$. Show that $r_{\alpha_i}(\gamma) = \gamma - \frac{2\alpha_i}{(\alpha_i,\alpha_i)}$.
Hint: use HW 16.1(a).

(b) Show that $\sum\limits_{\beta\in\Delta^+} \frac{(\alpha_i,\beta)}{(\beta,\beta)} = 1$.

(c) Let $v \in \mathbb{R}^n$, $v = \sum c_i\alpha_i$. Show that $\sum c_i = \sum\limits_{\beta\in\Delta^+} \frac{(v,\beta)}{(\beta,\beta)}$.

(d) Define quadratic from Q on $\mathbb{R}^n$ by $Q(v) = \sum\limits_{\beta\in\Delta^+} \frac{(v,\beta)^2}{(\beta,\beta)}$. Show that Q is invariant with respect to W. *Hint:* $Q(v) = \sum\limits_{\beta\in\Delta^+} \frac{(v,\beta)^2}{(\beta,\beta)} = \frac{1}{2}\sum\limits_{\beta\in\Delta} \frac{(v,\beta)^2}{(\beta,\beta)}$.

(e) Let $\{e_i\}$ be an orthonormal basis of $\mathbb{R}^n$. Denote $N = |\Delta^+|$. Show that $\sum\limits_{i=1}^{n}\sum\limits_{\beta\in\Delta^+} \frac{(e_i,\beta)^2}{(\beta,\beta)} = N$.

(f) Show that $\sum\limits_{\beta\in\Delta^+} \frac{(v,\beta)^2}{(\beta,\beta)} = (v,v)\frac{N}{n}$. Deduce from this that $\sum\limits_{\beta\in\Delta^+} \frac{(v,\beta)^2}{(v,v)(\beta,\beta)} = \frac{N}{n}$.
Hint: use HW 18.4.

(g) Let $\alpha, \beta \in \Delta$, and let $(\alpha,\alpha) \le (\beta,\beta)$. Show that $\langle \alpha \mid \beta\rangle = 0$ or ± 1.

(h) Show that $\langle \alpha \mid \tilde\alpha_0\rangle = \langle \alpha \mid \tilde\alpha_0\rangle^2$ for any positive root $\alpha \neq \tilde\alpha_0$.

(i) Show that $N = \dfrac{(\operatorname{ht}\tilde\alpha_0 + 1)n}{2}$. Deduce from this that $h = 1 + \operatorname{ht}\tilde\alpha_0$.
Hint: write $\frac{(\tilde\alpha_0,\beta)}{(\beta,\beta)}$ as $\langle\beta \mid \tilde\alpha_0\rangle\frac{(\tilde\alpha_0,\tilde\alpha_0)}{2(\beta,\beta)}$ and use (c),(f) and (h).

Solution:

(a)

$$r_{\alpha_i}(\gamma) = \sum_{\beta\in\Delta^+} \frac{r_{\alpha_i}(\beta)}{(\beta,\beta)} = \frac{r_{\alpha_i}(\alpha_i)}{(\alpha_i,\alpha_i)} + \sum_{\substack{\beta\in\Delta^+\\ \beta\neq\alpha_i}} \frac{r_{\alpha_i}(\beta)}{(\beta,\beta)} = -\frac{\alpha_i}{(\alpha_i,\alpha_i)} + \sum_{\substack{\beta'\in\Delta^+\\ \beta'\neq\alpha_i}} \frac{\beta'}{(\beta',\beta')} = \gamma - \frac{2\alpha_i}{(\alpha_i,\alpha_i)}$$

Here we used that r_{α_i} takes $\Delta^+\setminus\alpha_i$ to $\Delta^+\setminus\alpha_i$ (HW 16.1(a)), and that $\beta' = r_{\alpha_i}(\beta)$ has the same length as β.

(b) $\sum\limits_{\beta\in\Delta^+} \frac{(\alpha_i,\beta)}{(\beta,\beta)} = (\alpha_i,\gamma)$, which is equal to 1 by (a).

(c) This immediately follows from (b) by linearity.

(d) It is sufficient to verify the statement for generators of W, i.e. for r_{α_i}. Following the hint, we have

$$Q(r_{\alpha_i}(v)) = \frac{1}{2}\sum_{\beta\in\Delta}\frac{(r_{\alpha_i}(v),\beta)^2}{(\beta,\beta)} = \frac{1}{2}\sum_{\beta\in\Delta}\frac{(v,r_{\alpha_i}(\beta))^2}{(r_{\alpha_i}(\beta),r_{\alpha_i}(\beta))} = \frac{1}{2}\sum_{\beta'\in\Delta}\frac{(v,\beta')^2}{(\beta',\beta')} = Q(v)$$

(e)

$$\sum_{i=1}^{n}\sum_{\beta\in\Delta^+}\frac{(e_i,\beta)^2}{(\beta,\beta)} = \sum_{\beta\in\Delta^+}\sum_{i=1}^{n}\frac{(e_i,\beta)^2}{(\beta,\beta)} = \sum_{\beta\in\Delta^+}\frac{\|\beta\|^2}{(\beta,\beta)} = \sum_{\beta\in\Delta^+}1 = N$$

(f) According to (d), the quadratic form $Q(v) = \sum\limits_{\beta\in\Delta^+}\frac{(v,\beta)^2}{(\beta,\beta)}$ is invariant with respect to W. By HW 18.4, this implies that $Q(v) = c(v,v)$. By (e), we have $\sum\limits_{i=1}^{n}Q(e_i) = N$. Therefore, $N = \sum\limits_{i=1}^{n}Q(e_i) = \sum\limits_{i=1}^{n}c(e_i,e_i) = nc$, and thus $c = \frac{N}{n}$.

(g) This follows from Lemma 9.3: both $\langle\alpha\mid\beta\rangle$ and $\langle\beta\mid\alpha\rangle$ are integers and the modulus of their product does not exceed 3, so either both are zero or one of them must equal ±1.

(h) Since $(\tilde\alpha_0,\alpha_j)\geq 0$, (a) and HW 18.3 imply that $\langle\alpha_j\mid\tilde\alpha_0\rangle = 0$ or 1, and the statement follows.

(i) Following the hint, we write

$$\begin{aligned}\operatorname{ht}\tilde\alpha_0 &\overset{\text{by (c)}}{=} \sum_{\beta\in\Delta^+}\frac{(\tilde\alpha_0,\beta)}{(\beta,\beta)} = \sum_{\beta\in\Delta^+}\langle\beta\mid\tilde\alpha_0\rangle\frac{(\tilde\alpha_0,\tilde\alpha_0)}{2(\beta,\beta)} \overset{\text{by (h)}}{=}\\ &\overset{\text{by (h)}}{=} \sum_{\beta\in\Delta^+}\langle\beta\mid\tilde\alpha_0\rangle^2\frac{(\tilde\alpha_0,\tilde\alpha_0)}{2(\beta,\beta)} - \langle\tilde\alpha_0\mid\tilde\alpha_0\rangle^2\frac{(\tilde\alpha_0,\tilde\alpha_0)}{2(\tilde\alpha_0,\tilde\alpha_0)} + \langle\tilde\alpha_0\mid\tilde\alpha_0\rangle\frac{(\tilde\alpha_0,\tilde\alpha_0)}{2(\tilde\alpha_0,\tilde\alpha_0)} =\\ &= \sum_{\beta\in\Delta^+}\frac{4(\beta,\tilde\alpha_0)^2}{(\tilde\alpha_0,\tilde\alpha_0)^2}\frac{(\tilde\alpha_0,\tilde\alpha_0)}{2(\beta,\beta)} - 2 + 1 = 2\sum_{\beta\in\Delta^+}\frac{(\beta,\tilde\alpha_0)^2}{(\tilde\alpha_0,\tilde\alpha_0)(\beta,\beta)} - 1 \overset{\text{by (f)}}{=} 2\frac{N}{n} - 1,\end{aligned}$$

which implies $N = \frac{(\operatorname{ht}\tilde\alpha_0+1)n}{2}$. By Lemma 11.17(2), $N = \frac{hn}{2}$, so $h = \operatorname{ht}\tilde\alpha_0 + 1$.