Durham University
Pavel Tumarkin

Epiphany 2025

Topics in Combinatorics IV, Homework 19 (Week 19)

Throughout the problem sheet Δ is a root system of rank n, $\Pi = \{\alpha_i\}$ are simple roots, $\tilde{\alpha}_0$ is the highest root, W is the Weyl group, h is the Coxeter number.

19.1. Compute the Coxeter number and exponents of Weyl group of type

(a) C_4;

(b) C_n.

19.2. (a) Show that the Coxeter number of Weyl group of type E_8 is equal to the Coxeter number of the Coxeter group of type H_4.

(b) Show that the symmetric group S_{n+1} contains a subgroup isomorphic to the dihedral group $I_2(n+1)$.

(c) Let $W = \langle s_1, \ldots, s_4 \mid s_i^2, (s_2 s_j)^3 \text{ for } j \neq 2, (s_k s_l)^2 \text{ for } k, l \neq 2 \rangle$ be the Weyl group of type D_4. Show that the subgroup of W generated by s_1, s_2 and $s_3 s_4$ is isomorphic to the Weyl group of type B_3.

19.3. (a) Define $\gamma = \sum\limits_{\beta \in \Delta^+} \frac{\beta}{(\beta,\beta)}$. Show that $r_{\alpha_i}(\gamma) = \gamma - \frac{2\alpha_i}{(\alpha_i,\alpha_i)}$.

Hint: use HW 16.1(a).

(b) Show that $\sum\limits_{\beta \in \Delta^+} \frac{(\alpha_i,\beta)}{(\beta,\beta)} = 1$.

(c) Let $v \in \mathbb{R}^n$, $v = \sum c_i \alpha_i$. Show that $\sum c_i = \sum\limits_{\beta \in \Delta^+} \frac{(v,\beta)}{(\beta,\beta)}$.

(d) Define quadratic from Q on $\mathbb{R}^n$ by $Q(v) = \sum\limits_{\beta \in \Delta^+} \frac{(v,\beta)^2}{(\beta,\beta)}$. Show that Q is invariant with respect to W.

Hint: $Q(v) = \sum\limits_{\beta \in \Delta^+} \frac{(v,\beta)^2}{(\beta,\beta)} = \frac{1}{2} \sum\limits_{\beta \in \Delta} \frac{(v,\beta)^2}{(\beta,\beta)}$.

(e) Let $\{e_i\}$ be an orthonormal basis of $\mathbb{R}^n$. Denote $N = |\Delta^+|$. Show that $\sum\limits_{i=1}^{n} \sum\limits_{\beta \in \Delta^+} \frac{(e_i,\beta)^2}{(\beta,\beta)} = N$.

(f) Show that $\sum\limits_{\beta \in \Delta^+} \frac{(v,\beta)^2}{(\beta,\beta)} = (v,v)\frac{N}{n}$. Deduce from this that $\sum\limits_{\beta \in \Delta^+} \frac{(v,\beta)^2}{(v,v)(\beta,\beta)} = \frac{N}{n}$.

Hint: use HW 18.4.

(g) Let $\alpha, \beta \in \Delta$, and let $(\alpha,\alpha) \leq (\beta,\beta)$. Show that $\langle \alpha \mid \beta \rangle = 0$ or ± 1.

(h) Show that $\langle \alpha \mid \tilde{\alpha}_0 \rangle = \langle \alpha \mid \tilde{\alpha}_0 \rangle^2$ for any positive root $\alpha \neq \tilde{\alpha}_0$.

(i) Show that $N = \dfrac{(\operatorname{ht} \tilde{\alpha}_0 + 1)n}{2}$. Deduce from this that $h = 1 + \operatorname{ht} \tilde{\alpha}_0$.

Hint: write $\frac{(\tilde{\alpha}_0,\beta)}{(\beta,\beta)}$ as $\langle \beta \mid \tilde{\alpha}_0 \rangle \frac{(\tilde{\alpha}_0,\tilde{\alpha}_0)}{2(\beta,\beta)}$ and use (c),(f) and (h).