Durham University
Pavel Tumarkin

Michaelmas 2022

Topics in Combinatorics IV, Solutions 2 (Week 2)

2.1. Let n be a positive integer, and let p be a prime.

(a) Show that the number of sequences of integers $n_1, \dots, n_p$, where $1 \le n_i \le n$ and at least two n_i's are distinct, is equal to $n^p - n$.

(b) Show that all cyclic shifts of any sequence from (a) are distinct.

(c) Deduce that $n^p - n$ is divisible by p.

Solution:

(a) The total number of all sequences in n letters is n^p, the number of constant sequences is n.

(b) Since p is prime, powers of any cyclic shift exhaust all cyclic shifts of a given sequence. Thus, if any two cyclic shifts coincide, then all cyclic shifts of a given sequence coincide, and thus the sequence is constant, which is not the case by the assumption.

(c) According to (b), the set of all sequences is subdivided into equivalence classes of size p, where two sequences are equivalent if they are related by a cyclic shift. Therefore, $n^p - n$ is divisible by p.

2.2. Consider a Drunkard's walk in the segment $[0, n]$, i.e.:

- the walk starts at interger $x = i$, $0 \le i \le n$;
- the probability of steps left and right is equal to $1/2$;
- the walk ends when the drunkard reaches either $x = 0$ or $x = n$.

Denote by p_i the probability the walk starting at $x = i$ ends at point $x = n$.

(a) Show that $p_i = \frac{1}{2}p_{i-1} + \frac{1}{2}p_{i+1}$ for every $i = 1, \dots, n-1$.

(b) Compute p_i for every i.
Hint: you may need to recall some linear algebra.

(c) Deduce from (b) the result of Example 1.15 (Drunkard's walk) from lectures.

Solution:

(a) The first step leads either to $i+1$ or to $i-1$ with equal probability, so the result follows.

(b) After adding equations $p_0 = 0$ and $p_n = 1$, (a) leads to a system of $n+1$ linear equations in $n+1$ variables. Using Gaussian elimination, it is easy to see that $p_{k+1}/p_k = (k+1)/k$ for all $k > 0$. Now, in view of $p_n = 1$, we obtain $p_i = i/n$.

(c) The result of (b) can be understood as follows: the probability of the drunkard reaching 0 before n is equal $1 - \frac{i}{n}$. Taking the limit as $n \to \infty$, we see that the probability tends to 1 (note that in Example 1.15 we deduced this for $i = 1$ only).

2.3. Find a bijection between the set of non-decreasing sequences $1 \le a_1 \le \cdots \le a_n$ such that $a_i \le i$ and lattice paths in the $n \times n$ square from $(n, 0)$ to $(0, n)$ lying above the main diagonal (and thus, show that the number of such sequences is C_n).

Solution: A lattice path subdivides the square into two parts ("lower left" one and "upper right" one), and is uniquely defined by this subdivision. The upper part, in its turn, is uniquely defined by the height of all of its columns. Let a_i be the height of the column between $x = i - 1$ and $x = i$ plus one. Then the sequence (a_i) satisfies the assumptions.

Conversely, for every such sequence one can easily construct the corresponding lattice path, so we get a bijection.

2.4. ($\star$) We say that a Dyck path has a *hill* at point $2i + 1$ if it passes through points $(2i, 0)$ and $(2i + 2, 0)$. Denote by F_k the number of *hill-free* Dyck paths of length $2k$, i.e. Dyck paths without hills.

(a) Compute F_k for $k \le 5$.

(b) Show that numbers F_k satisfy the following equation:

$$C_n = F_n + \sum_{k=0}^{n-1} F_k C_{n-k-1},$$

where C_k are Catalan numbers.
Hint: consider the first hill from the left.

(c) Compute the generating function $F(x)$ of the sequence (F_k). Show that

$$F(x) = \frac{1}{1 - x^2 C(x)^2},$$

where $C(x)$ is the generating function for Catalan numbers.

Solution:

(a) $F_1 = 0$, $F_2 = 1$, $F_3 = 2$, $F_4 = 6$: in all these cases the first two steps must go up, the last two steps must go down, and between them one can take any lattice path, so the answer is $F_k = \binom{2k-4}{k-2}$. For $k = 5$, the first two and last steps still leave some freedom: there is one lattice path with a hill and one going below x-axis. Therefore, $F_5 = \binom{6}{3} - 2 = 18$.

(b) The reasoning is very similar to the one proving the Catalan recurrence. If there are no hills, we get F_n. Otherwise, considering the leftmost hill at point $2k+1$, we see that is subdivides the Dyck path into three parts: an arbitrary hill-free path of length $2k$, the hill, and an arbitrary Dyck path of length $2n - 2k - 2$. Thus, the number of Dyck paths with the leftmost hill at point $2k + 1$ is equal to $F_k C_{n-k-1}$. Taking the sum over all k, we obtain

$$C_n = F_n + \sum_{k=0}^{n-1} F_k C_{n-k-1}$$

as required.

(c) Again, we proceed as in the proof of Lemma 1.12.
Multiplying the equality in (b) by x^n, we get

$$C_n x^n = F_n x^n + \sum_{k=0}^{n-1} F_k C_{n-k-1} x^n = F_n x^n + x \sum_{k=0}^{n-1} (F_k x^k)(C_{n-k-1} x^{n-k-1}).$$

Now, summing on $n > 0$, we get on the left $C(x) - 1$. On the right, we get

$$\sum_{n=1}^{\infty} \left(F_n x^n + x \sum_{k=0}^{n-1} (F_k x^k)(C_{n-k-1} x^{n-k-1}) \right) =$$
$$= F(x) - 1 + x \sum_{m=0}^{\infty} \sum_{k=0}^{m} (F_k x^k)(C_{m-k} x^{m-k}) = F(x) - 1 + x \cdot F(x) \cdot C(x),$$

so we obtain the equation $C(x) - 1 = F(x) - 1 + xF(x)C(x)$, which implies $C(x) = F(x) + xF(x)C(x)$, and thus

$$F(x) = \frac{C(x)}{1 + xC(x)}.$$

Multiplying both numerator and denominator by $1 - xC(x)$, we obtain

$$F(x) = \frac{C(x)(1 - xC(x))}{(1 + xC(x))(1 - xC(x))} = \frac{1}{1 - x^2 C(x)^2}$$

since $C(x)(1 - xC(x)) = 1$.