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Topics in Combinatorics IV, Solutions 3 (Week 3)

3.1. ($\star$) Denote by $p_k(n)$ the number of Young diagrams $\lambda \vdash n$ with k rows. Show that

$$p_1(n) + p_2(n) + \cdots + p_k(n) = p_k(n+k)$$

Solution: Use induction on k. If $k = 1$, then $p_1(n) = p_1(n+1) = 1$. Assume that the statement holds for k. We need to prove that $p_k(n+k) + p_{k+1}(n) = p_{k+1}(n+k+1)$. Given a Young diagram $\lambda \vdash n+k+1$ with $k+1$ rows, we have a dichotomy: either the last row is of length one, or not. In the former case, by removing the last row we get a Young diagram $\lambda' \vdash n+k$ with k rows, so the number of such diagrams is $p_k(n+k)$. In the latter case, we can remove one box from every row to get a Young diagram $\lambda'' \vdash n$ with $k+1$ rows, so the number of such diagrams is $p_{k+1}(n)$. Thus, we constructed a bijection of a set of size $p_{k+1}(n+k+1)$ to a set of size $p_k(n+k) + p_{k+1}(n)$.

Alternatively, one can construct a direct bijection between partitions of $n+k$ into k blocks, and partitions of n into at most k blocks. Take any Young diagram $\lambda \vdash n+k$ with k rows, and remove the first column (consisting of k boxes), we get a partition of n into at most k blocks. This map is clearly injective and surjective.

3.2. ($\star$) A partition (or a Young diagram) $\lambda = (\lambda_1, \dots \lambda_k) \vdash n$ is called *self-conjugate* if the height of i-th column of λ is equal to λ_i for every i. Show that the number of self-conjugate partitions of n is equal to the number of partitions of n with all summands odd and distinct.

Solution: Given $\lambda \vdash n$, consider all boxes of type (i,i) in λ (let the number of them be m), and the sequence $h(1,1), \dots, h(m,m)$ of hook lengths. Then all elements of the sequence are odd and distinct (the sequence is strictly decreasing), their sum is equal to n, and clearly the sequence is uniquely defined by λ. So, we constructed an injective map from the set of self-conjugate Young diagrams to required partitions. Conversely, given a partition of n with all summands odd and distinct, place them in decreasing order and construct a self-conjugate partition of n with these hook lengths. The requirement for the numbers to decrease guarantees that we will get a Young diagram (as $\lambda_i = i + (h(i,i)-1)/2 \le \lambda_{i-1}$ for $i \le m$, while for $i > m$ the property $\lambda_i \le \lambda_{i-1}$ follows from self-conjugacy).

Given a polygon, a *dissection* of it is a collection of mutually non-crossing diagonals (e.g., triangulation is an example of a dissection).

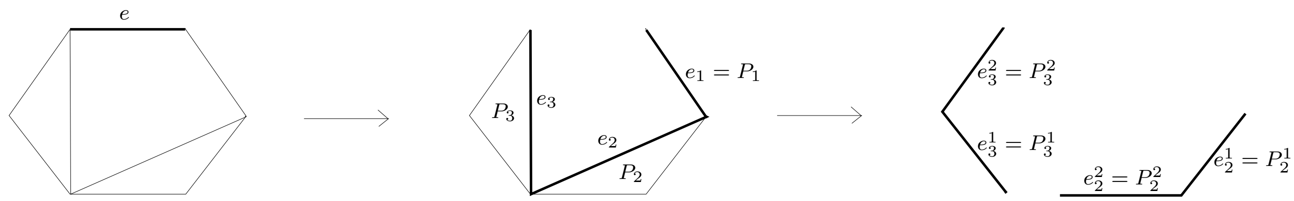
Given a convex $(n+2)$-gon P with one marked edge e, and a dissection of P with d diagonals, define a sequence $\varphi(P) = (a_1, \dots, a_m)$ of integers recursively as follows.

Mark all edges $e = e_0, e_1, \dots, e_k$ of the smallest polygon of the dissection containing e clockwise. By removing the edge e we obtain a sequence of dissected polygons $P_1, \dots, P_k$ (where

P_i has an edge e_i), such that P_i and P_{i+1} have a unique common vertex (note that some of P_i may consist of a single edge e_i). Then define

$\varphi(P) = (k-1, \varphi(P_1), -1, \varphi(P_2), -1, \ldots, \varphi(P_{k-1}), -1, \varphi(P_k))$, where $\varphi(P_i = e_i) = \emptyset$.

Example. $n = 4$, $d = 2$



$\varphi(P) = (2, -1, \varphi(P_2), -1, \varphi(P_3))$ $\qquad$ $\varphi(P) = (2, -1, 1, -1, -1, 1, -1)$

3.3. (a) Show that the resulting sequence $\varphi(P) = (a_1, \ldots, a_m)$ satisfies the following properties:

- $m = n + d + 1$;
- $a_i \in \mathbb{Z}$, $a_i = -1$ or $a_i > 0$ for every $i = 1, \ldots, n + d + 1$;
- the number of negative ones is precisely n;
- $\sum\limits_{i=1}^{n+d+1} a_i = 0$;
- $\sum\limits_{i=1}^{l} a_i \geq 0$ for every positive integer $l \leq n + d + 1$.

Hint: use induction on n and d.

(b) Show that a partial sum of $\varphi(P)$ vanishes if and only if the dissection contains a diagonal incident to the common vertex of e and its counterclockwise neighboring edge.

(c) Show that every sequence characterized by five properties in (a) can be obtained as $\varphi(P)$ for some dissection of P with d diagonals. Show that the map φ establishes a bijection between the set of dissections of an $(n+2)$-gon with d diagonals and the set of sequences characterized by five properties in (a).

Solution:

(a) The proof is by induction on n and d. The base is $n = 1$, $d = 0$, i.e. an empty dissection of a triangle. Then $n + d + 1 = 2$, and the sequence is $(1, -1)$ which clearly satisfies all the requirements.

Now assume we have $\varphi(P) = (k-1, \varphi(P_1), -1, \varphi(P_2), -1, \ldots, \varphi(P_{k-1}), -1, \varphi(P_k))$, where out of k polygons P_i precisely l are single edges e_i. Then we have the following.

- Precisely $k - l$ edges e_i are diagonals, and $d - k + l$ diagonals are used in dissections of smaller polygons P_i (where we consider $k - l$ polygons $P_i \neq e_i$). Let P_i have $n_i + 2$ sides and d_i diagonals inside. Then, by the induction assumption, the length of $\varphi(P_i)$ is $n_i + d_i + 1$, and the length of $\varphi(P)$ is equal to

$$k + \sum_{P_i \neq e_i} (n_i + d_i + 1) = k + \sum_{P_i \neq e_i} (n_i + 2) + \sum_{P_i \neq e_i} d_i + (k - l) - 2(k - l) =$$
$$= l + \sum_{P_i \neq e_i} (n_i + 2) + \sum_{P_i \neq e_i} d_i = l + ((n + 1 - l) + (k - l)) + (d - k + l) = n + d + 1,$$

since the total number of sides of polygons P_i is $(n + 1 - l) + (k - l)$.

- $a_i \in \mathbb{Z}$, $a_i = -1$ or $a_i > 0$ for every $i = 1, \dots, n+d+1$ by construction;
- note that there is a bijection between positive entries and domains of the dissection, while the number of domains is equal to $d+1$, so the number of negative ones is $(n+d+1) - (d+1) = n$; this can also be proved by induction;
- by the induction assumption, the sum of entries in every $\varphi(P_i)$ is zero; at the last step, we add entry $a_1 = (k-1)$ and precisely $k-1$ negative ones, so the sum remains zero;
- any partial sum of $\varphi(P)$ consists of a sum of $k-1$, several $\varphi(P_i)$, several negative ones (the number of which does not exceed $k-1$), and, probably, a partial sum of some $\varphi(P_i)$; by the induction assumption, every partial sum of entries in every $\varphi(P_i)$ is non-negative, so it follows this is also true for $\varphi(P)$.

(b) This follows from the reasoning above: for a partial sum to vanish it must contain all $k-1$ negative ones inserted at the last step, and thus the last $\varphi(P_k)$ should not be empty.

(c) We need to reconstruct a dissection by a sequence, and to show that there is a unique way to do this – this would provide the required inverse map.

We use induction on the length of the sequence. The base is $n+d+1 = 2$, then we immediately get a triangle.

Now consider a sequence of length $n+d+1$. Assume that some partial sum of the sequence vanishes, say $a_1 + \dots + a_l = 0$. By the induction assumption, the sequence $(a_1, \dots, a_l)$ is obtained from a dissection of an $(m+2)$-gon P_1 with $l-m-1$ diagonals, where m is equal to the number of negative ones in $(a_1, \dots, a_l)$ and thus is reconstructed uniquely. Then the remaining part $(a_{l+1}, \dots, a_{n+d+1})$ of the sequence is also obtained from a dissection of a polygon P_2 with $(n-m+2)$ sides and $(n+d+1-l)-(n-m)-1 = d+m-l$ diagonals. Gluing these two polygons along the "last" side of P_1 and the "first" side of P_2 results in a dissection of a polygon P which has $(m+2)+(n-m+2)-2 = n+2$ sides and $(l-m-1)+(d+m-l)+1 = d$ diagonals.

Assume now that no partial sum is equal to zero, i.e. every partial sum is positive. This implies that the sequence $(a_1 - 1, a_2, a_3, \dots, a_{n+d})$ has all partial sums non-negative, and thus is obtained from a dissection of an $(n+1)$-gon P' with d diagonals. Now the original sequence corresponds to a dissection of an $(n+2)$-gon P which is obtained from P' by adding an additional edge between e and its counterclockwise neighboring side.

Let $d < n$ be non-negative integers, consider a Young diagram $\lambda = (d+1, d+1, 1, \dots 1) \vdash n+d+1$ (i.e., there are $n-d-1$ of ones). Given a sequence $(a_1, \dots, a_{n+d+1})$ as in Problem 3.3(a), we will now construct a SYT of shape λ by inserting numbers $1, \dots, n+d+1$ into λ in turn.

Denote by $b_1, \dots, b_{d+1}$ all positive elements of the sequence, $b_i = a_{m_i}$, $m_i < m_j$ for $i < j$. Then the rules for inserting numbers are the following.

- If $a_i > 0$, then i is inserted at the end of the first row (i.e., directly to the right of all elements which are already in the first row);
- if $a_i = -1$ and the number of -1's preceding a_i is of the form $b_1 + \dots + b_j$ for some $j \geq 0$ then i inserted at the end of the second row;
- otherwise, i is inserted at the bottom of the first column.

3.4. (a) Show that the outcome of the procedure above is indeed a SYT of shape λ;

(b) Show that the construction above establishes a bijection between the set of SYT of shape λ and the set of sequences characterized in Problem 3.3(a).

Solution:

(a) There are $d+1$ positive entries, so $\lambda_1 = d+1$. There are $d+2$ partial sums $b_1 + \cdots + b_j$, $j \geq 0$, but the sum of all positive entries equals n, and thus there is no -1 such that the number of preceding negative ones is equal to $b_1 + \cdots + b_{d+1}$, so $\lambda_2 = d+1$ as required. Rows and the first column are monotonic by construction. We only need to ensure other columns are monotonic.

Indeed, the entry in the box $(1, l)$ is i where $a_i = b_l$. The entry j in the box $(2, l)$ has $b_1 + \cdots + b_{l-1}$ preceding negative ones. Thus, if $j < i$, then the part $(a_1, \ldots, a_j)$ of the sequence contains at least $b_1 + \cdots + b_{l-1} + 1$ negative ones and at most $l-1$ positive numbers, so the partial sum $a_1 + \cdots + a_j$ is negative.

(b) Given a SYT, we can uniquely reconstruct the sequence as follows (note that we need to know the places and values of $d+1$ positive entries only).

The first row gives the places of all positive values $b_1, \ldots, b_{d+1}$. The entry j_1 in box $(2, 1)$ is equal to the smallest number not included in the first row. Now let the entry in box $(2, 2)$ be equal to j_2. This means that the number of negative ones preceding j_2 is equal to b_1. Since we know all entries of the first row, we do know how many positive numbers precede j_2, so we do know how many negative numbers precede j_2, and thus we recover b_1. Similarly, j_3 allows us to recover $b_1 + b_2$, and thus b_2. Continuing, we recover all b_i, $i \leq d$. Then $b_{d+1} = n - (b_1 + \cdots + b_d)$.

3.5. Use the hook length formula and Problems 3.3 and 3.4 to show that the number of dissections of an $(n+2)$-gon with d diagonals is equal to

$$\frac{1}{n+d+2}\binom{n+d+2}{d+1}\binom{n-1}{d}$$

Solution: The hook lengths in the first row are $(n+1, d+1, d, \ldots, 2)$, in the second row $(n, d, d-1, \ldots, 1)$, and in the remaining part of the first column $(n-1-d, n-2-d, \ldots, 1)$. The hook length formula implies that the number of SYT is equal to

$$f_\lambda = \frac{(n+d+1)!}{(n+1)(d+1)!nd!(n-1-d)!} = \frac{(n-1)!}{d!(n-1-d)!}\frac{(n+d+1)!}{(d+1)!(n-1)!n(n+1)} =$$
$$= \frac{(n-1)!}{d!(n-1-d)!}\frac{(n+d+2)!}{(d+1)!(n+1)!}\frac{1}{n+d+2} = \binom{n-1}{d}\binom{n+d+2}{d+1}\frac{1}{n+d+2}$$