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Topics in Combinatorics IV, Homework 3 (Week 3)

Due date for starred problems: **Friday, November 3, 6pm**.

3.1. ($\star$) Denote by $p_k(n)$ the number of Young diagrams $\lambda \vdash n$ with k rows. Show that $p_1(n) + p_2(n) + \cdots + p_k(n) = p_k(n+k)$

3.2. ($\star$) A partition (or a Young diagram) $\lambda = (\lambda_1, \dots \lambda_k) \vdash n$ is called *self-conjugate* if the height of i-th column of λ is equal to λ_i for every i. Show that the number of self-conjugate partitions of n is equal to the number of partitions of n with all summands odd and distinct.

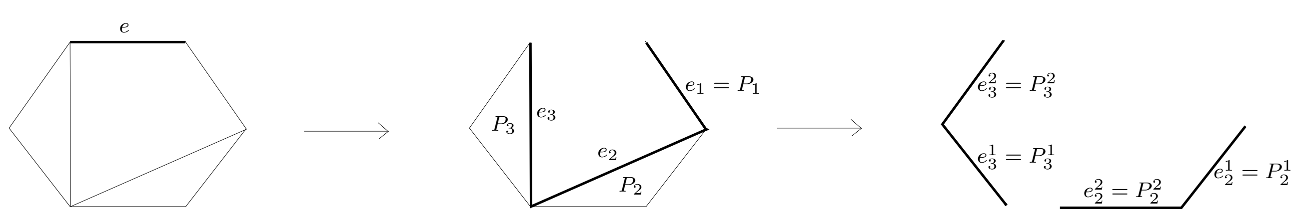
Given a polygon, a *dissection* of it is a collection of mutually non-crossing diagonals (e.g., triangulation is an example of a dissection).

Given a convex $(n+2)$-gon P with one marked edge e, and a dissection of P with d diagonals, define a sequence $\varphi(P) = (a_1, \dots, a_m)$ of integers recursively as follows.

Mark all edges $e = e_0, e_1, \dots, e_k$ of the smallest polygon of the dissection containing e clockwise. By removing the edge e we obtain a sequence of dissected polygons $P_1, \dots, P_k$ (where P_i has an edge e_i), such that P_i and P_{i+1} have a unique common vertex (note that some of P_i may consist of a sigle edge e_i). Then define

$\varphi(P) = (k-1, \varphi(P_1), -1, \varphi(P_2), -1, \dots, \varphi(P_{k-1}), -1, \varphi(P_k))$, where $\varphi(P_i = e_i) = \emptyset$.

Example. $n = 4$, $d = 2$



$\varphi(P) = (2, -1, \varphi(P_2), -1, \varphi(P_3))$ $\qquad$ $\varphi(P) = (2, -1, 1, -1, -1, 1, -1)$

3.3. (a) Show that the resulting sequence $\varphi(P) = (a_1, \dots, a_m)$ satisfies the following properties:

- $m = n + d + 1$;
- $a_i \in \mathbb{Z}$, $a_i = -1$ or $a_i > 0$ for every $i = 1, \dots, n+d+1$;
- the number of negative ones is precisely n;
- $\displaystyle\sum_{i=1}^{n+d+1} a_i = 0$;
- $\displaystyle\sum_{i=1}^{l} a_i \geq 0$ for every positive integer $l \leq n + d + 1$.

Hint: use induction on n and d.

(b) Show that some proper partial sum of $\varphi(P)$ vanishes if and only if the dissection contains a diagonal incident to the common vertex of e and its counterclockwise neighboring edge.

(c) Show that every sequence characterized by five properties in (a) can be obtained as $\varphi(P)$ for some dissection of P with d diagonals. Show that the map φ establishes a bijection between the set of dissections of an $(n+2)$-gon with d diagonals and the set of sequences characterized by five properties in (a).

Let $d < n$ be non-negative integers, consider a Young diagram $\lambda = (d+1, d+1, 1, \dots 1) \vdash n+d+1$ (i.e., there are $n-d-1$ of ones). Given a sequence $(a_1, \dots, a_{n+d+1})$ as in Problem 3.3(a), we will now construct a SYT of shape λ by inserting numbers $1, \dots, n+d+1$ into λ in turn.

Denote by $b_1, \dots, b_{d+1}$ all positive elements of the sequence, $b_i = a_{m_i}$, $m_i < m_j$ for $i < j$. Then the rules for inserting numbers are the following.

- If $a_i > 0$, then i is inserted at the end of the first row (i.e., directly to the right of all elements which are already in the first row);
- if $a_i = -1$ and the number of -1's preceding a_i is of the form $b_1 + \dots + b_j$ for some $j \geq 0$ then i inserted at the end of the second row;
- otherwise, i is inserted at the bottom of the first column.

3.4. (a) Show that the outcome of the procedure above is indeed a SYT of shape λ;

(b) Show that the construction above establishes a bijection between the set of SYT of shape λ and the set of sequences characterized in Problem 3.3(a).

3.5. Use the hook length formula and Problems 3.3 and 3.4 to show that the number of dissections of an $(n+2)$-gon with d diagonals is equal to

$$\frac{1}{n+d+2}\binom{n+d+2}{d+1}\binom{n-1}{d}$$