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Topics in Combinatorics IV, Solutions 4 (Week 4)

4.1. Construct a bijection between all set partitions of $[n]$ and those set partitions of $[n+1]$ that do not contain consequent numbers in one block.

Solution: A rook placement of a set partition of $[n+1]$ that does not contain consequent numbers in one block is characterized by the following: it has no rooks in the lowest diagonal. Removing the empty diagonal, we obtain a rook placement for a set partition of $[n]$. The inverse map is just an addition of an empty diagonal.

4.2. ($\star$) Show that the number of non-crossing set partitions of $[n]$ with k blocks is equal to the Narayana number $N(n,k)$.
Hint: you may try different models of Catalan numbers we considered in lectures.

Solution: Recall that we have a bijection between Dyck paths of length $2n$ and rooted plane trees with n edges. Observe that the map gives a bijection between trees with k leaves and Dyck paths with k peaks.

Thus, we need to find a bijection between non-crossing partitions of $[n]$ with k blocks and rooted plane trees with n edges and k leaves.

Take a tree, index the non-root vertices in the order you meet them when go around the tree counterclockwise (recall the construction of the bijection). Now go around the tree clockwise, and take as a block the set of vertices you meet before every leaf (including the leaf itself). We get a partition with k blocks; it is non-crossing since the tree can be considered as a "gluing of a circle" with numbers $1,\dots,n$ written on it, while preimages of edges do not intersect. (Alternatively, one can argue as follows. Consider any block B, and assume it contains p,q and does not contain l where $p<l<q$. We may assume that $p=\max\{i \mid i<l, i\in B\}$, and $q=\min\{j \mid j>l, j\in B\}$. Then, by construction, the nodes $p+1,\dots,q-1$ form a separate subtree, and thus are not involved in any blocks with other nodes.)

Two different trees give rise to distinct partitions, and as we know that the total number is C_n in both cases, it follows that we get all non-crossing partitions with k blocks.

4.3. Show the symmetry of Narayana numbers: $N(n,k)=N(n,n-k+1)$.

Solution:

According to the previous exercise, we need to find a bijection between plane trees with n edges and k leaves and plane trees with n edges and $n-k+1$ leaves.

First introduce the notation: for every edge, the vertex on the top is a *parent*, and the vertex at the bottom is a *child*. Children of the same parent are *siblings*, and there is also a natural definition of *ancestors* – these include the vertex itself and the direct line of parents. A *right sibling* is a sibling which you meet after meeting the parent while going counterclockwise around the tree.

Now, the bijection works as follows. We keep the vertices but change the edges. For any non-root vertex, find the closest ancestor that has a right sibling (it can be the vertex itself if it has one). Then, if there is no such ancestor, the new parent is the root. If there is such ancestor, the new parent is the right sibling of that ancestor.

4.4. A *star graph* is a graph whose all vertices except for one are leaves (i.e., it consists of one vertex connected to every other vertex).

(a) Let c_n be the number of star graphs on n labeled nodes (the graph is not embedded, i.e. it only matters which vertex is connected to which). Compute c_n for every $n \geq 1$.

(b) Show that the exponential generating function $c(x)$ of the sequence (c_n) is

$$c(x) = xe^x - \frac{x^2}{2}$$

Solution:

(a) The graph is completely defined by the only "central" vertex, i.e. the number of such graphs is n with one exception: $c_2 = 1$.

(b) We have

$$c(x) = \frac{x}{1} + \frac{x^2}{2!} + \frac{3x^3}{3!} + \frac{4x^4}{4!} + \cdots = \sum_{n\geq 1} \frac{nx^n}{n!} - \frac{x^2}{2} = \sum_{n\geq 1} \frac{x^n}{(n-1)!} - \frac{x^2}{2} = x \sum_{m\geq 0} \frac{x^m}{m!} - \frac{x^2}{2} = xe^x - \frac{x^2}{2}$$