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Topics in Combinatorics IV, Solutions 5 (Week 5)

5.1. Let c_n denote the number of c-objects on n labeled nodes (as in the lectures), $n \geq 1$. Denote by $d_{n,k}$ the number of d-objects on n nodes with k components, i.e. the number of collections of k c-objects with total number of nodes being n (e.g., $d_{n,1} = c_n$, and $\sum\limits_k d_{n,k} = d_n$). Define

$$d(x,y) = \sum_{n\geq 0}\sum_{k\geq 0} d_{n,k}\frac{x^n}{n!}y^k$$

Show that $d(x,y) = e^{y\cdot c(x)}$, where $c(x)$ is the exponential generating function of (c_n).

Solution: As we proved at the lectures, the number of d-object with fixed k and blocks of sizes $n_1,\dots,n_k$ is equal to

$$\frac{1}{k!}\binom{n}{n_1\, n_2\, \dots\, n_k} c_{n_1}\dots c_{n_k}.$$

To get $d_{n,k}$, we need to take a sum over $n_1+\dots+n_k = n$:

$$d_{n,k} = \sum_{\substack{n_1+\dots+n_k=n\\ n_1,\dots,n_k\geq 1}} \frac{1}{k!}\frac{n!}{n_1!\,n_2!\,\dots\,n_k!}c_{n_1}\dots c_{n_k},$$

which implies that

$$\frac{d_{n,k}}{n!} = \sum_{\substack{n_1+\dots+n_k=n\\ n_1,\dots,n_k\geq 1}} \frac{1}{k!}\frac{c_{n_1}}{n_1!}\frac{c_{n_2}}{n_2!}\cdots\frac{c_{n_k}}{n_k!},$$

and thus

$$d(x,y) = \sum_{n\geq 0}\sum_{k\geq 0} d_{n,k}\frac{x^n}{n!}y^k = \sum_{n\geq 0}\sum_{k\geq 0}\sum_{\substack{n_1+\dots+n_k=n\\ n_1,\dots,n_k\geq 1}} \frac{y^k}{k!}\frac{c_{n_1}}{n_1!}\frac{c_{n_2}}{n_2!}\cdots\frac{c_{n_k}}{n_k!}x^n =$$

$$= \sum_{k\geq 0}\frac{y^k}{k!}\sum_{n_1,\dots,n_k\geq 1}\frac{c_{n_1}x^{n_1}}{n_1!}\frac{c_{n_2}x^{n_2}}{n_2!}\cdots\frac{c_{n_k}x^{n_k}}{n_k!} = \sum_{k\geq 0}\frac{y^k}{k!}\left(\sum_{m\geq 1}\frac{c_m x^m}{m!}\right)^k = \sum_{k\geq 0}\frac{y^k}{k!}c(x)^k = e^{y\cdot c(x)}.$$

5.2. Recall that Stirling number of second kind $S(n,k)$ is defined as the number of set partitions of $[n]$ into k blocks.

(a) Show that $\sum\limits_{n,k\geq 0} S(n,k)\frac{x^n}{n!}y^k = e^{y(e^x-1)}$.

(b) Prove the following recurrence relation:

$$S(n,k) = S(n-1,k-1) + kS(n-1,k)$$

Solution:

(a) This is an immediate corollary of Problem 5.1: let c-objects be sets, and d-objects be set partitions.

(b) Take any set partition of $[n]$ into k blocks, and consider the place of n. There are $S(n-1,k-1)$ partitions with n forming a single block. Further, if n is not alone in its block, we can remove it and get a partition of $[n-1]$ into k blocks. Any such partition can be obtained precisely k times (as n could be in any block), so the result follows.

5.3. ($\star$) Define the *falling factorial* $y_{(k)} = y(y-1)\dots(y-k+1) = k!\binom{y}{k}$ for any $y \in \mathbb{R}$.

(a) Show that the number of surjective functions $f : [n] \to [k]$ is equal to $S(n,k) \cdot k!$.

(b) Show that for any $m, n \in \mathbb{N}$

$$\sum_{k=0}^{n} \binom{m}{k} S(n,k) \cdot k! = m^n$$

(c) Show that

$$\sum_{k=0}^{n} S(n,k) y_{(k)} = y^n$$

Solution:

(a) Denote $B_i = f^{-1}(i)$, then a function defines a set partition of $[n]$ into blocks $B_1, \dots, B_k$, where different order of blocks leads to different fuctions. Therefore, the number of functions is equal to $k!$ times the number of set partitions with k blocks, i.e. $k! \cdot S(n,k)$.

(b) RHS is the number of all functions from $[n]$ to $[m]$. In the LHS the factor $\binom{m}{k}$ counts all k-subsets of $[m]$, so $\binom{m}{k}S(n,k) \cdot k!$ is the number of all surjective functions of $[n]$ onto k-element subsets of $[m]$. Every function from $[n]$ to $[m]$ is surjective onto its image (whose cardinality does not exceed n), so by taking sum over k from 0 to n we count all functions, as in the RHS.

(c) Both sides of the equality are polynomials in y. In every integer point $y = m$ the LHS is equal to

$$\sum_{k=0}^{n} S(n,k) m_{(k)} = \sum_{k=0}^{n} S(n,k) \binom{m}{k} \cdot k! = m^n$$

according to (b). Thus, the values of polynomials coincide at every integer point, and therefore they are equal to each other.

5.4. Define the *signless Stirling number of the first kind* $c(n,k)$ as the number of permutations $w \in S_n$ with $\operatorname{cyc}(w) = k$, and *Stirling number of the first kind* as $s(n,k) = (-1)^{n-k} c(n,k)$. We define $c(0,0) = 1$ and $c(n,0) = c(0,n) = 0$ for $n > 0$.

(a) Show that $c(n,k) = c(n-1,k-1) + (n-1)c(n-1,k)$.

(b) Define the *raising factorial* $y^{(k)} = y(y+1)\dots(y+k-1) = k!\binom{y+k-1}{k}$ for any $y \in \mathbb{R}$. Show that

$$\sum_{k=0}^{n} c(n,k)x^k = x^{(n)}$$

(c) Show that

$$\sum_{k=0}^{n} s(n,k)x^k = x_{(n)}$$

Solution:

(a) The solution is identical to the one of Problem 5.2(b), the only difference is that every $w \in S_{n-1}$ can be obtained precisely $n-1$ times by removing n from the cyclic notation of permutations from S_n.

(b) We use induction on n. For $n = 1$ the identity becomes $c(1,1)x = x$, which holds since there is a unique permutation on one letter with one cycle.

Now, using (a) and the induction assumption, we can write

$$\sum_{k=0}^{n} c(n,k)x^k = x\sum_{k=1}^{n} c(n-1,k-1)x^{k-1} + \sum_{k=0}^{n}(n-1)c(n-1,k)x^k =$$
$$= x\sum_{i=0}^{n-1} c(n-1,i)x^i + (n-1)\sum_{k=0}^{n} c(n-1,k)x^k = x \cdot x^{(n-1)} + (n-1)x^{(n-1)} = x^{(n)}$$

(c) (a) implies that $s(n,k) = s(n-1,k-1) - (n-1)s(n-1,k)$. Now, proceeding as in (b), we obtain

$$\sum_{k=0}^{n} s(n,k)x^k = x\sum_{k=1}^{n} s(n-1,k-1)x^{k-1} - \sum_{k=0}^{n}(n-1)s(n-1,k)x^k =$$
$$= x\sum_{i=0}^{n-1} s(n-1,i)x^i - (n-1)\sum_{k=0}^{n} s(n-1,k)x^k = x \cdot x_{(n-1)} - (n-1)x_{(n-1)} = x_{(n)}$$