Durham University
Pavel Tumarkin

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Topics in Combinatorics IV, Solutions 6 (Week 6)

6.1. Recall that given $w \in S_n$, $\text{exc}\,(w)$ is the number of excedances of w (i.e. places $i \in [n]$ such that $i < w_i$).

Complete the proof of Theorem 3.13: show that statistics des and exc are equidistributed.

Solution: Define *anti-excedance* of w as $i \in [n]$ such that $i > w_i$. Observe, that anti-excedances of w are precisely excedances of w^{-1}, so anti-excedances and excedances are equidistributed.

Now, we claim that the map f from the proof of Thm 3.13 (which takes a standard form of cycle decomposition and produces a 1-line form of a permutation by erasing all brackets) takes anti-excedances of w to descents of $w' = f(w)$.

Indeed, anti-excedance in the standard form of cycle decomposition consists of two consequent entries $b_i b_{i+1}$ such that $b_i > b_{i+1}$ (note that in the standard form the first element of the cycle is the maximal entry in the whole cycle, so the last entry cannot be an anti-excedance). After erasing the brackets, all anti-excedeances become descents by definition. The only other possibilities for descents could be the last entries of cycles, but the condition that the first elements of cycles increase guarantees this does not happen. Therefore, the map takes anti-excedances to all descents, and thus anti-excedances and descents are equidistributed, so the result follows.

6.2. Let $w = w_1\, w_2\, \dots\, w_n \in S_n$, $n \geq 2$. $i \in [n]$ is a *weak excedance* of w if $w_i \geq i$. Denote by $\text{wexc}\,(w)$ the number of weak excedances of $w \in S_n$.

Show that statistics exc and wexc $-\,1$ are equidistributed.

Solution:

Let $w = w_1\, w_2\, \dots\, w_n \in S_n$ have $k+1$ weak excedances. Then the permutation w^{-1} has $n-(k+1) = (n-1)-k$ excedances: weak excedance of w is $i \leq w_i$, and excedance of w^{-1} is $w_i < i$, which are clearly complementary.

Now, we already know that exc and des are equidistributed, and there is a bijection $f : S_n \to S_n$ taking excedances to descents. Take $f(w^{-1}) = v = v_1\, v_2\, \dots\, v_n$ having $(n-1)-k$ descents, and observe that $v' = v_n\, v_{n-1}\, \dots\, v_1$ has $(n-1)-k$ ascents and k descents (as for every descent $v_i < v_{i+1}$ there is an ascent in v', and vice versa). Now, take $f^{-1}(v)$: this permutation has precisely k excedances.

Therefore, we constructed a bijection $S_n \to S_n$ defined by $w \mapsto w^{-1} \mapsto v = f(w^{-1}) \mapsto v' \mapsto f^{-1}(v')$ which takes a permutation with $k+1$ weak excedances to a permutation with k excedances.

6.3. ($\star$) Define *Eulerian numbers* $A(n,k)$ as the numbers of permutations $w \in S_n$ with $\text{des}\,(w) = k-1$, $k \leq n$.

Show that $A(n, k+1) = (n-k)A(n-1, k) + (k+1)A(n-1, k+1)$.

Solution:

The solution is similar to the one of Problems 5.2(b) and 5.4(a).

Take any permutation $w \in S_{n-1}$ with l descents, and insert n at some place. Denote by $w' \in S_n$ the resulting permutation. There are $l+1$ positions of n such that $\operatorname{des}(w) = \operatorname{des}(w') = l$, these correspond to placing n after descents of w or at the very end; for all the others $\operatorname{des}(w') = \operatorname{des}(w) + 1 = l+1$, there are $n-(l+1) = n - \operatorname{des}(w) - 1$ such places. Thus, we can obtain a permutation in S_n with k descents either by inserting n into $(n-k)$ places of $A(n-1,k)$ permutations in S_{n-1} with $(k-1)$ descents (here $n-k = n - \operatorname{des}(w) - 1 = n-(k-1)-1$), or by inserting n into $(k+1)$ places of $A(n-1,k+1)$ permutations in S_{n-1} with k descents. As every permutation from the corresponding set can be obtained, the result follows.

6.4. ($\star$) Let P_1, P_2 be posets. A map $f : P_1 \to P_2$ is called *order-preserving* if for any $a, b \in P_1$ the relation $a \leq_{P_1} b$ implies $f(a) \leq_{P_2} f(b)$.

(a) Let P be a finite poset, and let $f : P \to P$ be an order-preserving bijection. Show that f^{-1} is also order-preserving.

(b) Show that for infinite posets the statement of part (a) may not hold.

Solution:

(a) Since P is finite, the bijection f is a permutation of elements of P belonging to $S_{|P|}$, and thus f has a finite order (say, $k \in \mathbb{N}$). Therefore, $f^{-1} = f^{k-1}$ for some natural $k > 1$, so it is a composition of order-preserving maps, and hence it is order-preserving itself.

(b) Consider $P = \mathbb{Z} \cup \{t\}$, where $t > z_0 \in \mathbb{Z}$ and t is incomparable with all $z > z_0$ for some given z_0. Then $f : P \to P$ defined by $f(z) = z-1$, $f(t) = t$ is a bijection of P preserving the order, but the inverse is not (as $z_0 + 1 = f^{-1}(z_0)$ and $t = f^{-1}(t)$ are incomparable, while $z_0 < t$).