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Topics in Combinatorics IV, Solutions 7 (Week 7)

7.1. Define a *lattice* axiomatically as a set L with two binary operations $\vee$ and $\wedge$ satisfying the following properties:

- Reflexive law: $x \vee x = x \wedge x = x$;
- Commutative law: $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$;
- Associative law: $(x \vee y) \vee z = x \vee (y \vee z)$ and $(x \wedge y) \wedge z = x \wedge (y \wedge z)$;
- Absorption law: $x \vee (x \wedge y) = x$ and $x \wedge (x \vee y) = x$.

Show that this axiomatic definition of lattice is equivalent to the one from lectures: a lattice is a poset such that for any two elements meet and join exist.

Solution: First, assume we are given a poset with join and meet. Reflexivity clearly holds, and the definition of join and meet are symmetric with respect to two elements, so commutativity also follows. Associativity follows from the fact both join and meet can be defined for more than two elements, and both sides of the equality coincide with the join (or meet) for three elements. Finally, absorption is implied by the following two properties: $x \wedge y \leq x$, and $z \vee x = x$ for $z \leq x$ (and the same for swapping signs and join with meet). Thus, all axioms hold.

Conversely, if all axioms hold, define the order on L by $x \leq y$ if $x \wedge y = x$. We need to check this is indeed an order. Symmetry is obvious (follows from commutativity), reflexivity follows directly from axioms. To prove transitivity , we need to show that $x \wedge y = x$ and $y \wedge z = y$ implies $x \wedge z = x$. Indeed, by associativity $x \wedge z = (x \wedge y) \wedge z = x \wedge (y \wedge z) = x \wedge y = x$.

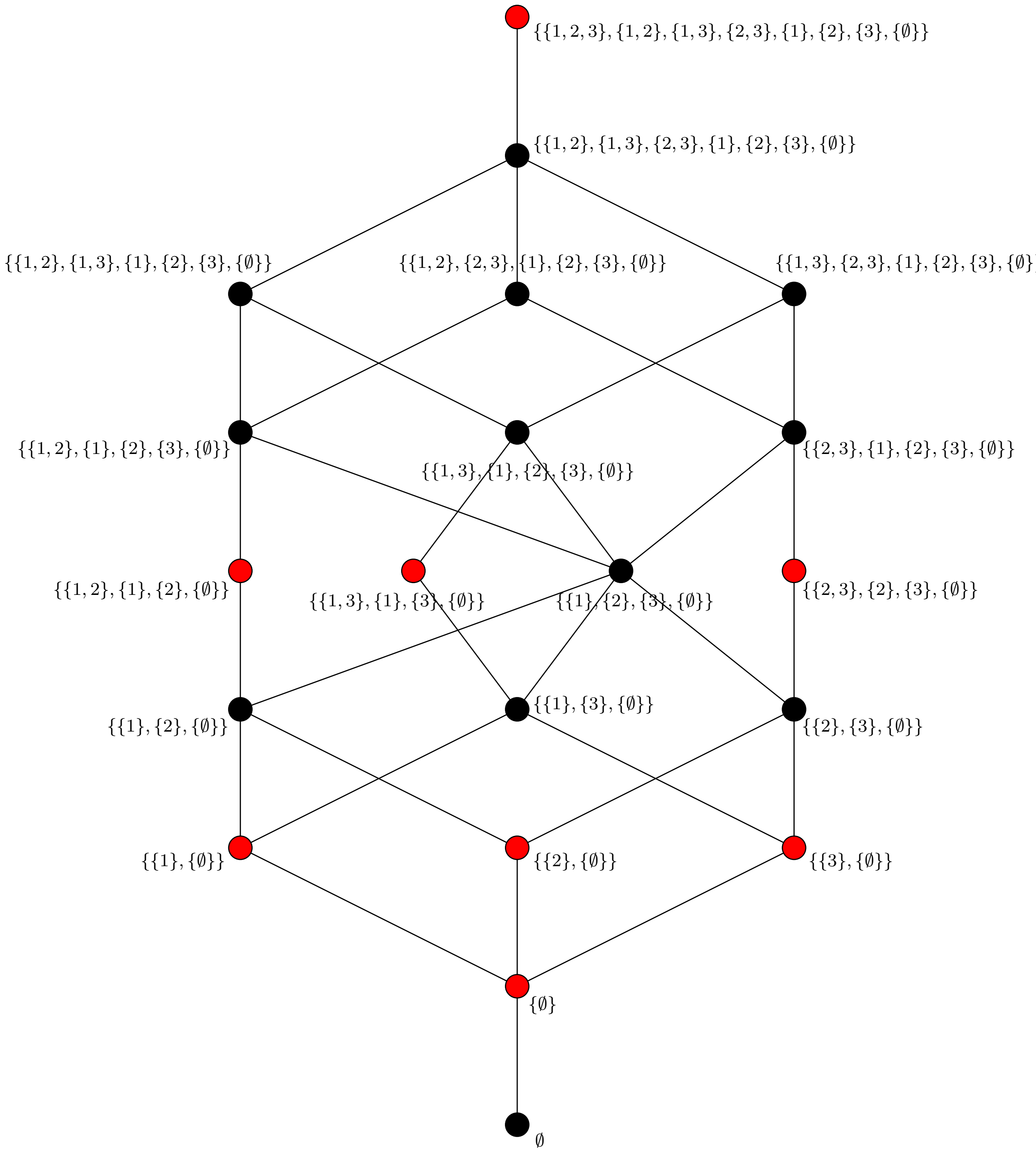
We now need to show that if $z \leq x, y$, then $z \leq x \wedge y$. This is equivalent to the following: if $z \wedge x = z$ and $z \wedge y = z$, then $z \wedge (x \wedge y) = z$. Indeed, $z \wedge (x \wedge y) = (z \wedge x) \wedge y = z \wedge y = z$ by associativity.

We will now show that $z \geq x, y$ implies $z \geq x \vee y$. For this we will prove that $x \wedge y = x$ implies $x \vee y = y$ (and thus vice versa by symmetry): this follows from the absorption law, since if $x \wedge y = x$, then $y = y \vee (y \wedge x) = y \vee x = x \vee y$. Now all statements for join can be recovered from the ones for meet by replacing meet to join and changing the sign.

7.2. ($\star$) Draw the Hasse diagram of the poset of order ideals of the Boolean lattice B_3 (identifying elements at every vertex). Identify join-irreducible elements of $J(B_3)$. (The latter is actually a hint.)

Solution:

The lattice $J(B_3)$ is ranked, where the rank is the number of elements in an order ideal, which ranges from 0 to 8. One can find the elements in every rank by adding one element of B_3 to elements of previous rank and checking whether this is an order ideal. The result is shown in the picture below (the join-irreducible elements are red, and it is clear they form a B_3 lattice).



7.3. ($\star$) Show that the set Π_n of set partitions of $[n]$ ordered by refinement is a lattice. Is it distributive?

Solution:

Define meet of λ and μ as the partition with blocks being intersections of all possible pairs of blocks of λ and μ (and then omit empty sets). Then $\lambda \wedge \mu \leq \lambda, \mu$, and if $\nu \leq \lambda \wedge \mu$, then every block of ν is contained in some block of $\lambda \wedge \mu$, and thus is contained in some blocks of λ and μ.

Now define a join of λ and μ as follows. Take arc diagrams of λ and μ and draw them simultaneously on $[n]$. The blocks of $\lambda \vee \mu$ are connected components of the resulting arc diagram. Clearly, $\lambda, \mu \leq \lambda \vee \mu$, and if $\nu \geq \lambda, \mu$ then $\nu \geq \lambda \vee \mu$.

Π_n is not distributive for $n \geq 3$ as already Π_3 is not distributive. For example, consider $\lambda = (1\,2 \mid 3)$, $\mu = (1 \mid 2\,3)$, $\nu = (2 \mid 1\,3)$. Then

$$\lambda \vee (\mu \wedge \nu) = (1\,2 \mid 3) \vee (1 \mid 2 \mid 3) = (1\,2 \mid 3),$$

while

$$(\lambda \vee \mu) \wedge (\lambda \vee \nu) = (1\,2\,3) \wedge (1\,2\,3) = (1\,2\,3).$$

7.4. Let P be a poset such that every chain and every antichain is finite. Show that P is finite. *Hint:* consider the set of minimal elements of P.

Solution:

Consider the set P_0 of minimal elements of P , and define inductively P_k to be the set of minimal elements of $P \setminus \bigcup_{i<k} P_i$. Every P_k is an antichain, so by the assumption it is finite. Assume that P is infinite, then there are infinitely many such antichains.

For every $t \in P$ there is $p \in P_0$ such that $p \leq t$, so there exists $p_0 \in P_0$ such that $P^{(1)} \stackrel{\text{def}}{=} \{t \in P \mid t \geq p_0\}$ is infinite. Consider minimal $k > 0$ such that $P^{(1)} \cap P_k \neq \emptyset$. There exists $p_1 \in P_k$ such that $P^{(2)} \stackrel{\text{def}}{=} \{t \in P^{(1)} \mid t \geq p_1\}$ is infinite. Continuing in the same way, we obtain an infinite chain.