Durham University
Pavel Tumarkin

Michaelmas 2023

Topics in Combinatorics IV, Homework 7 (Week 7)

Due date for starred problems: **Friday, December 1, 6pm**.

7.1. Define a *lattice* axiomatically as a set L with two binary operations $\vee$ and $\wedge$ satisfying the following properties:

- Reflexive law: $x \vee x = x \wedge x = x$;
- Commutative law: $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$;
- Associative law: $(x \vee y) \vee z = x \vee (y \vee z)$ and $(x \wedge y) \wedge z = x \wedge (y \wedge z)$;
- Absorption law: $x \vee (x \wedge y) = x$ and $x \wedge (x \vee y) = x$.

Show that this axiomatic definition of lattice is equivalent to the one from lectures: a lattice is a poset such that for any two elements meet and join exist.

7.2. ($\star$) Draw the Hasse diagram of the poset of order ideals of the Boolean lattice B_3 (identifying elements at every vertex). Identify join-irreducible elements of $J(B_3)$. (The latter is actually a hint.)

7.3. ($\star$) Show that the set Π_n of set partitions of $[n]$ ordered by refinement is a lattice. Is it distributive?

7.4. Let P be a poset such that every chain and every antichain is finite. Show that P is finite. *Hint:* consider the set of minimal elements of P.