Topics in Combinatorics IV, Solutions 8 (Week 8)

8.1. Show that the poset J(P) of order ideals of a poset P is a distributive lattice.

Solution: First, we need to check that the union and intersection of order ideals is again an order ideal. This allow us to define $I \wedge J = I \cap J$ and $I \vee J = I \cup J$.

Indeed, if $y \in I \cap J$, then $y \in I, J$. If $x \in P$, $x \leq y$, then $x \in I, J$ as well, so $x \in I \cap J$. Now, assume $y \in I \cup J$, and $x \leq y$. Without loss of generality, we may assume $y \in I$. But then $x \in I$ as $x \leq y$, so $x \in I \subset I \cup J$.

The rest of the proof follows from the fact B_n is a distributive lattice: unions and intersections of sets satisfy both distributive laws.

8.2. Complete the proof of Lemma 4.30. Given a poset P with |P| = n, construct a map from the set of linear extensions of P to the set of saturated chains of J(P) by taking $\varphi : P \to [n]$ to the chain $\hat{0} = \emptyset < I_1 < I_2 < \dots < I_n = \hat{1}$, where $I_j = \varphi^{-1}([j])$. Show that this map is a bijection.

Solution: The map given in the assumptions is clearly order-preserving, so we just need to construct the inverse map. Assume there is a saturated chain $\emptyset < I_1 < I_2 < \dots < I_n$ in J(P). Since there are precisely *n* non-empty ideals in the chain and they all must have distinct cardinalities, we have $|I_j| = j$. Now define $\varphi(j) = I_j \setminus I_{j-1}$ (where $I_0 = \emptyset$). This map is clearly inverse to the one given in the assumptions.

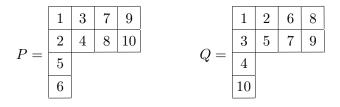
8.3. (*) Let $w = 26514871093 \in S_{10}$. Apply the RSK algorithm to w to obtain SYT P and Q.

Solution:

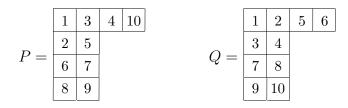
- Step 1: 2 is inserted in the box (1, 1).
- Step 2: 6 is inserted in the box (1, 2).
- Step 3: 5 is inserted in the box (1, 2), and 6 is pushed down into the box (2, 1).
- Step 4: 1 is inserted in the box (1,1), and thus pushes down 2 into the second row, where it goes to the box (2,1), and 6 is pushed down into the box (3,1).
- Step 5: 4 is inserted in the box (1, 2), and thus pushes down 5 into the second row, where 5 forms a new box (2, 2).
- Step 6: 8 is inserted in the box (1,3).
- Step 7: 7 is inserted in the box (1,3), and thus pushes down 8 into the second row, where 8 forms a new box (2,3).
- Step 8: 10 is inserted in the box (1, 4).

- Step 9: 9 is inserted in the box (1, 4), and thus pushes down 10 into the second row, where 10 forms a new box (2, 4).
- Step 10: 3 is inserted in the box (1, 2), and thus pushes down 4 into the second row, where it goes to the box (2, 2), and 5 is pushed down to the third row, where it goes to the box (3, 1), and 6 is pushed down to the forth row into the box (4, 1).

As a result, we get



8.4. (*) Let (P,Q) be SYT of shape $\lambda = (4,2,2,2) \vdash 10$, where



Construct $w \in S_{10}$ which is taken to the pair (P, Q) by the RSK algorithm.

Solution:

- Step 10: box (4, 2). 9 is pushed down by 7 (which is the maximal element in row 3 which is less than 9), which is pushed down by 5, which is pushed down by 4, so $w_{10} = 4$.
- Step 9: box (4, 1). 8 is pushed down by 6, which is pushed down by 2, which is pushed down by 1, so $w_9 = 1$.
- Step 8: box (3,2). 9 is pushed down by 7, which is pushed down by 5, so $w_8 = 5$.
- Step 7: box (3,1). 8 is pushed down by 6, which is pushed down by 3, so $w_7 = 3$.
- Step 6: box (1, 4). $w_6 = 10$.
- Step 5: box (1,3). $w_5 = 7$.
- Step 4: box (2, 2). 9 is pushed down by 6, so $w_4 = 6$.
- Step 3: box (2, 1). 8 is pushed down by 2, so $w_3 = 2$.
- Step 2: box (1, 2). $w_2 = 9$.
- Step 1: box (1, 1). $w_1 = 8$.

Therefore, w = 89267103514.