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Topics in Combinatorics IV, Solutions 8 (Week 8)

8.1. Show that the poset $J(P)$ of order ideals of a poset P is a distributive lattice.

Solution: First, we need to check that the union and intersection of order ideals is again an order ideal. This allow us to define $I \wedge J = I \cap J$ and $I \vee J = I \cup J$.

Indeed, if $y \in I \cap J$, then $y \in I, J$. If $x \in P$, $x \leq y$, then $x \in I, J$ as well, so $x \in I \cap J$. Now, assume $y \in I \cup J$, and $x \leq y$. Without loss of generality, we may assume $y \in I$. But then $x \in I$ as $x \leq y$, so $x \in I \subset I \cup J$.

The rest of the proof follows from the fact B_n is a distributive lattice: unions and intersections of sets satisfy both distributive laws.

8.2. Complete the proof of Lemma 4.30. Given a poset P with $|P| = n$, construct a map from the set of linear extensions of P to the set of saturated chains of $J(P)$ by taking $\varphi : P \to [n]$ to the chain $\hat{0} = \emptyset \lessdot I_1 \lessdot I_2 \lessdot \ldots \lessdot I_n = \hat{1}$, where $I_j = \varphi^{-1}([j])$. Show that this map is a bijection.

Solution: The map given in the assumptions is clearly order-preserving, so we just need to construct the inverse map. Assume there is a saturated chain $\emptyset \lessdot I_1 \lessdot I_2 \lessdot \ldots \lessdot I_n$ in $J(P)$. Since there are precisely n non-empty ideals in the chain and they all must have distinct cardinalities, we have $|I_j| = j$. Now define $\varphi(j) = I_j \setminus I_{j-1}$ (where $I_0 = \emptyset$). This map is clearly inverse to the one given in the assumptions.

8.3. ($\star$) Let $w = 2\,6\,5\,1\,4\,8\,7\,10\,9\,3 \in S_{10}$. Apply the RSK algorithm to w to obtain SYT P and Q.

Solution:

Step 1: 2 is inserted in the box $(1,1)$.

Step 2: 6 is inserted in the box $(1,2)$.

Step 3: 5 is inserted in the box $(1,2)$, and 6 is pushed down into the box $(2,1)$.

Step 4: 1 is inserted in the box $(1,1)$, and thus pushes down 2 into the second row, where it goes to the box $(2,1)$, and 6 is pushed down into the box $(3,1)$.

Step 5: 4 is inserted in the box $(1,2)$, and thus pushes down 5 into the second row, where 5 forms a new box $(2,2)$.

Step 6: 8 is inserted in the box $(1,3)$.

Step 7: 7 is inserted in the box $(1,3)$, and thus pushes down 8 into the second row, where 8 forms a new box $(2,3)$.

Step 8: 10 is inserted in the box $(1,4)$.

Step 9: 9 is inserted in the box $(1,4)$, and thus pushes down 10 into the second row, where 10 forms a new box $(2,4)$.

Step 10: 3 is inserted in the box $(1,2)$, and thus pushes down 4 into the second row, where it goes to the box $(2,2)$, and 5 is pushed down to the third row, where it goes to the box $(3,1)$, and 6 is pushed down to the forth row into the box $(4,1)$.

As a result, we get

$$P = \begin{ytableau} 1 & 3 & 7 & 9 \\ 2 & 4 & 8 & 10 \\ 5 \\ 6 \end{ytableau} \qquad Q = \begin{ytableau} 1 & 2 & 6 & 8 \\ 3 & 5 & 7 & 9 \\ 4 \\ 10 \end{ytableau}$$

8.4. $(\star)$ Let (P,Q) be SYT of shape $\lambda = (4,2,2,2) \vdash 10$, where

$$P = \begin{ytableau} 1 & 3 & 4 & 10 \\ 2 & 5 \\ 6 & 7 \\ 8 & 9 \end{ytableau} \qquad Q = \begin{ytableau} 1 & 2 & 5 & 6 \\ 3 & 4 \\ 7 & 8 \\ 9 & 10 \end{ytableau}$$

Construct $w \in S_{10}$ which is taken to the pair (P,Q) by the RSK algorithm.

Solution:

Step 10: box $(4,2)$. 9 is pushed down by 7 (which is the maximal element in row 3 which is less than 9), which is pushed down by 5, which is pushed down by 4, so $w_{10} = 4$.

Step 9: box $(4,1)$. 8 is pushed down by 6, which is pushed down by 2, which is pushed down by 1, so $w_9 = 1$.

Step 8: box $(3,2)$. 9 is pushed down by 7, which is pushed down by 5, so $w_8 = 5$.

Step 7: box $(3,1)$. 8 is pushed down by 6, which is pushed down by 3, so $w_7 = 3$.

Step 6: box $(1,4)$. $w_6 = 10$.

Step 5: box $(1,3)$. $w_5 = 7$.

Step 4: box $(2,2)$. 9 is pushed down by 6, so $w_4 = 6$.

Step 3: box $(2,1)$. 8 is pushed down by 2, so $w_3 = 2$.

Step 2: box $(1,2)$. $w_2 = 9$.

Step 1: box $(1,1)$. $w_1 = 8$.

Therefore, $w = 8\,9\,2\,6\,7\,10\,3\,5\,1\,4$.