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Topics in Combinatorics IV, Solutions 9 (Week 9)

9.1. Denote by $r_x(P)$ an insertion of x in a partial tableau P in the RSK algorithm. Suppose that during $r_x(P)$ the elements $x_1, \dots, x_k$ are pushed down from rows $1, 2, \dots, k$ and columns $j_1, j_2, \dots, j_k$ respectively. Then

(a) $x < x_1 < \cdots < x_k$;

(b) $j_1 \geq \cdots \geq j_k$;

(c) if $P' = r_x(P)$, then $P'_{i,j} \leq P_{i,j}$ for all i, j.

Solution:

(a) This follows directly from the definition of the algorithm: x_i pushes down x_{i+1} if $x_i < x_{i+1}$.

(b) Let x_i be pushed down from column j_i. Since P is a partial tableau, the element at the place $(i+1, j_i)$ is strictly larger than x_i, and thus x_i is inserted in the row $i+1$ in the column j_i or less, i.e. $j_{i+1} \leq j_i$.

(c) This follows from (a): if $P'_{i,j} \neq P_{i,j}$, then the element $P'_{i,j}$ was pushed down from row $i-1$, and $P_{i,j}$ is pushed down to row $i+1$, i.e. $P'_{i,j} = x_{i-1} < x_i = P_{i,j}$ in the notation of (a).

9.2. (a) Show that

$$\sum_{\lambda \vdash n} f_\lambda = \#\{w \in S_n \mid w^2 = 1\}$$

(b) Show that

$$\sum_{\lambda \vdash n} f_\lambda = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \frac{(2k)!}{2^k\, k!}$$

Solution:

(a) By Theorem 5.11, $w \in S_n$ is taken to (P, Q) if and only if w^1 is taken to (Q, P). Thus, $w = w^{-1}$ if and only if $P = Q$, so we got a bijection between involutions in S_n and SYT of total size n.

(b) By (a), we just need to count the number of permutations in S_n of order 2 or 1. These are precisely those which are decomposed into cycles of length 1 and 2.

Let an involution in S_n contain precisely k cycles of length 2. Clearly, k may vary from 0 to $\lfloor n/2 \rfloor$. To define such an involution we need first to choose $2k$ numbers involved in 2-cycles (there are $\binom{n}{2k}$ possibilities), and then to choose how we split $2k$ into k groups of 2 (the number of possibilities for the latter is given by the multinomial coefficient $\binom{2k}{2,2,\dots,2} = \frac{(2k)!}{2!\dots 2!} = \frac{(2k)!}{2^k}$. Note that the order of the 2-cycles is irrelevant (they all commute as they contain distinct numbers), so we also need to divide by $k!$. Therefore, for given k we get $\binom{n}{2k}\frac{(2k)!}{2^k\, k!}$ involutions with k 2-cycles, and thus the total number of involutions in S_n is equal to

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \frac{(2k)!}{2^k\, k!}$$

9.3. (A bit of linear algebra) Let A be a real symmetric indecomposable $n \times n$ matrix with all off-diagonal elements being non-positive. Show that A is positive definite if and only if there exists a vector $v \in \mathbb{R}^n$ with all positive coordinates such that all coordinates of Av are also positive.

Hint: use *Perron-Frobenius Theorem* which states that if all entries of a square indecomposable matrix are non-negative, then it has a simple positive eigenvalue μ such that μ has maximal modulus amongst all eigenvalues of A, and all the coordinates of the corresponding eigenvector are positive.

Solution:

Let m be the maximal positive diagonal value of a_{ij} (or zero if all diagonal element are negative), consider the matrix $A' = mI - A$. All elements of A' are non-negative, so, by the Perron-Frobenius theorem, there is an eigenvalue $\mu > 0$ of A' and eigenvector $\boldsymbol{v} = (v_1, \ldots, v_n)$ such that $A'\boldsymbol{v} = \mu\boldsymbol{v}$ and $v_i > 0$.

Observe that eigenvectors of A and A' coincide: if $A\boldsymbol{x} = \gamma\boldsymbol{x}$, then $A'\boldsymbol{x} = (mI - A)\boldsymbol{x} = (m - \gamma)\boldsymbol{x}$. In particular, if the eigenvalues of A' are $\mu = \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ (there are n real eigenvalues as A' is symmetric), then the eigenvalues of A are $m - \mu \leq m - \mu_2 \leq \cdots \leq m - \mu_n$, and thus $m - \mu$ is the smallest eigenvalue of A. If A is positive definite, then $m - \mu > 0$, and thus $\boldsymbol{v}$ satisfies the assumptions: all $v_i > 0$, and $(A\boldsymbol{v})_i = (m - \mu)v_i > 0$.

Therefore, we proved that if A is positive definite then there exists a vector $\boldsymbol{v} \in \mathbb{R}^n$ with all positive coordinates such that all coordinates of $A\boldsymbol{v}$ are also positive.

Now assume that such a vector exists (call it $\boldsymbol{h}$) and A is *not* positive definite. Let, as before, $\boldsymbol{v}$ be an eigenvector for the minimal eigenvalue of A (which is less or equal than zero), we can assume $\boldsymbol{v}$ has all coordinates positive as shown above. We are going to deduce a contradiction from the coexistence of $\boldsymbol{h}$ and $\boldsymbol{v}$.

The proof below follows the ideas of Theorem 6.1 (and, in particular, it also provides a proof of the implication (2)→(1) of Theorem 6.1). Of course, this is one of possible proofs only, there exist pure linear-algebraic proofs as well.

Since $m - \mu \leq 0$, $A\boldsymbol{v}$ has all coordinates negative or zero, and thus $(\boldsymbol{v}, \boldsymbol{A}_i) = v_i \leq 0$ for any i, where $\boldsymbol{A}_i$ is the i-th row of A.

Let $\boldsymbol{u} = \boldsymbol{u}_0 = c\boldsymbol{v}$ for $c > 0$ large enough (we will specify what c should be equal to later). Consider the following iterative process (similar to the Cartan firing game with initial configuration $\boldsymbol{u}$, see Section 6.2 of lectures): choose one i such that $(\boldsymbol{u}_k)_i \geq a_{ii}$, and define $\boldsymbol{u}_{k+1} = \boldsymbol{u}_k - \boldsymbol{A}_i$. Observe that we have the following:

$$(\boldsymbol{u}_{k+1}, \boldsymbol{h}) = (\boldsymbol{u}_k - \boldsymbol{A}_i, \boldsymbol{h}) = (\boldsymbol{u}_k, \boldsymbol{h}) - (\boldsymbol{A}_i, \boldsymbol{h}) \leq (\boldsymbol{u}_k, \boldsymbol{h}) - M,$$

where $M = \min_j(\boldsymbol{A}_j, \boldsymbol{h}) = \min_j(A\boldsymbol{h})_j > 0$. Therefore, after each step the inner product $(\boldsymbol{u}_k, \boldsymbol{h})$ decreases by at least $M > 0$, so after a sufficiently large number of steps it will become negative. On the other hand, all coordinates of $\boldsymbol{u}_k$ are non-negative for every k: this is true for $\boldsymbol{u}_0$, and the property is preserved at each step – if $\boldsymbol{u}_{k+1} = \boldsymbol{u}_k - \boldsymbol{A}_i$, then $(\boldsymbol{u}_{k+1})_i = (\boldsymbol{u}_k)_i - a_{ii} \geq 0$ by the choice of i, and $(\boldsymbol{u}_{k+1})_j = (\boldsymbol{u}_k)_j - a_{ij} \geq (\boldsymbol{u}_k)_j \geq 0$ for $i \neq j$ since all off-diagonal elements of A are non-positive. Thus, both $\boldsymbol{u}_k$ and $\boldsymbol{h}$ have all coordinates non-negative, and therefore $(\boldsymbol{u}_k, \boldsymbol{h}) \geq 0$ for all k. The contradiction shows that there exists k_0 such that $(\boldsymbol{u}_{k_0})_i < a_{ii}$ for all i, so the Cartan firing is finite. In particular, we get

$$(\boldsymbol{u}_{k_0}, \boldsymbol{u}) \leq n \cdot m \cdot \max_i\{(\boldsymbol{u})_i\} = cnm \max_i\{(\boldsymbol{v})_i\}.$$

Consider now $(\boldsymbol{u}_{k+1}, \boldsymbol{u})$. We have

$$(\boldsymbol{u}_{k+1}, \boldsymbol{u}) = (\boldsymbol{u}_k - \boldsymbol{A}_i, \boldsymbol{u}) = (\boldsymbol{u}_k, \boldsymbol{u}) - (\boldsymbol{A}_i, \boldsymbol{u}) \geq (\boldsymbol{u}_k, \boldsymbol{u})$$

since $(\boldsymbol{A}_i, \boldsymbol{u}) = c(\boldsymbol{A}_i, \boldsymbol{v}) \leq 0$. In other words, at every step the scalar product $(\boldsymbol{u}_k, \boldsymbol{u})$ (weakly) increases. In particular, we have $(\boldsymbol{u}_{k_0}, \boldsymbol{u}) \geq (\boldsymbol{u}, \boldsymbol{u})$. Combining this with the bound we obtained above, we get

$$cnm \max_i \{(\boldsymbol{v})_i\} \geq (\boldsymbol{u}_{k_0}, \boldsymbol{u}) \geq (\boldsymbol{u}, \boldsymbol{u}) = c^2(\boldsymbol{v}, \boldsymbol{v}).$$

Now, since $(\boldsymbol{v}, \boldsymbol{v}) > 0$, we can take c large enough so the above inequality does not hold, and thus we come to a contradiction.