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Topics in Combinatorics IV, Revision problems (Week 21)

These are examples from the second revision lecture. All HW problems are also revision problems.

R.1. Let $W = \langle s_1, \dots, s_4 \mid s_i^2, (s_2 s_j)^3 \text{ for } j \neq 2, (s_k s_l)^2 \text{ for } k, l \neq 2 \rangle$ be the Weyl group of type D_4. Show that the subgroup Γ of W generated by s_2 and $s_1 s_3 s_4$ is isomorphic to the dihedral group of type $G_2 = I_2(6)$.

Denote $a = s_2$, $b = s_1 s_3 s_4$. Then Γ is generated by a, b with some relations. Observe that $a^2 = b^2 = 1$, so all elements of Γ are alternating products of a and b. A words with an odd number of letters cannot be trivial (as it is conjugated to a or b), so the only missing relation is $(ab)^m$ for some m. Since $ab = s_2 s_1 s_3 s_4$ is a Coxeter element of W, its order is the Coxeter number of D_4, which can be easily found from the formula $N = nh/2$, where N is the number of positive roots, h is the Coxeter number, n is the dimension. Namely, $n = 4$, $N = 12$ as positive roots of D_4 are of the type $e_i \pm e_j$ (where $1 \leq i < j \leq 4$), so $h = 6$. Therefore,

$$\Gamma = \langle a, b \mid a^2, b^2, (ab)^6 \rangle$$

R.2. Let Δ be the root system of type D_5. Compute the Coxeter number of Δ. Find the exponents of the Weyl group of Δ.

The set of roots of Δ is $\{\pm e_i \pm e_j\}$, where $i, j = 1, \dots, 5$, $i < j$, and $\{e_i\}$ is an orthonormal basis of $\mathbb{R}^5$.

The number of positive roots of D_5 is $N = 20$, the rank is $n = 5$. Using the formula $h = 2N/n$ we see that $h = 8$.

Now, we can use the result from lectures stating that all integers from 1 to $h-1$ coprime with h are exponents. This implies that $1, 3, 5, 7$ are exponents. The sum of exponents is equal to $N = 20$, so the only missing exponent is 4. (Note that the last exponent can also be found as $h/2$ as it should provide the only real eigenvalue of the Coxeter element).

Alternatively, a linear map for a Coxeter element can be written explicitly. For example, for $c = r_{e_1-e_2} r_{e_2-e_3} r_{e_3-e_4} r_{e_4-e_5} r_{e_4+e_5}$ the matrix of c in the basis $\{e_i\}$ is

$$c = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

and it is easy to see that its order is 8.

The explicit expression for the matrix of a Coxeter element (see the alternative solution above) implies that the characteristic polynomial is $(-x-1)(x^4+1)$, so the eigenvalues are $-1 = e^{\frac{2\pi i}{8}4}$ and $e^{\frac{\pi i}{4}+\frac{2k\pi i}{4}} = e^{(1+2k)\frac{2\pi i}{8}}$, where $k = 0, 1, 2, 3$. Therefore, exponents are $1, 3, 4, 5, 7$.

Alternatively, one can use a result from lectures that the Young diagram $\lambda = (l_1, \ldots, l_k)$ which is conjugate to the Young diagram $\mu = (m_5, m_4, m_3, m_2, m_1)$ (where m_i are the exponents) satisfies the following: l_i is equal to the number of positive roots of height i. Thus, we are left to compute the number of positive roots of every height.

A root of type $e_i - e_j$ can be written as

$$e_i - e_j = \sum_{k=i}^{j-1}(e_k - e_{k+1}),$$

so the height of $e_i - e_j$ is $j - i$. This number takes value 1 four times, 2 three times, 3 two times, and 4 one time.

A root of type $e_i + e_j$ can be written as

$$e_i + e_j = (e_i - e_5) + (e_j + e_5) = (e_i - e_5) + (e_j - e_4) + (e_4 + e_5),$$

so the height of $e_i + e_j$ is $(5-i)+(4-j)+1 = 10-(i+j)$. Note that if $j = 5$ then the previous calculation should be adjusted, but the answer is still correct. The number $10-(i+j)$ takes values $1, 2, 6, 7$ one time each, and $3, 4, 5$ two times each.

Thus, we have $l_1 = 5$, $l_2 = l_3 = 4$, $l_4 = 3$, $l_5 = 2$, and $l_6 = l_7 = 1$. This implies that $\mu = (m_5, m_4, m_3, m_2, m_1) = (7, 5, 4, 3, 1)$.