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# Topics in Combinatorics IV, Problems Class 2 (Week 4)

**2.1.** Find a bijection between non-crossing matchings on $2n$ vertices and ballot sequences of length $2n$.

The bijection is the following: left side of an arc corresponds to $-1$, right side of an arc corresponds to $+1$. There is a place in the sequence where $+1$ is followed by $-1$, this corresponds to a short arc $i\,i+1$, delete it and reduce to the case of length $2(n-1)$.

**2.2.** Compute the number of SYT of shape $(n, n, n)$, find a combinatorial interpretation of these (i.e., find objects these SYT count).

The number can be found by the hook length formula: $H(\lambda) = \frac{(n+2)!}{2}(n+1)!n!$, so $f_\lambda = \frac{2(3n)!}{n!(n+1)!(n+2)!}$.

These SYT can be considered as "3-dim Dyck paths": lattice paths from $(0, 0, 0)$ to $(n, n, n)$ such that they lie in the tetrahedron with vertices $(0, 0, 0)$, $(n, 0, 0)$, $(n, n, 0)$ and $(n, n, n)$, i.e. they lie in the domain $\{(x, y, z) \mid 0 \le z \le y \le x \le n\}$.

**2.3.** Compute the number of "beginnings of all Dyck paths" between $(0, 0)$ and $(2n - k, k)$.

There are several ways to do this, one is by reflection: we get it to be the difference of all lattice paths between $(0, 0)$ and $(2n-k, k)$ and all lattice paths between $(0, 0)$ and $(2n-k, -k-2)$, i.e. the number is

$$\binom{2n-k}{n} - \binom{2n-k}{n+1} = \frac{k+1}{2n-k+1}\binom{2n-k+1}{n+1}$$