Topics in Combinatorics IV, Problems Class 5 (Week 12)

The class was devoted to the counterpart of question 11.1, where P is a regular tetrahedron in \mathbb{R}^3 centered at the origin.

- **5.1.** Let P be a regular tetrahedron in \mathbb{R}^3 centered at the origin. A symmetry of P is $g \in O_3(\mathbb{R})$ taking P to itself.
 - (a) Show that symmetries of P compose a group, denote it by Sym P. If $g_1, g_2 \in O_3(\mathbb{R})$ take P to itself, then so do g_i^{-1} and g_1g_2 .
 - (b) Show that Sym P acts on the set of faces of P transitively. Denote the vertices of P by v₁,..., v₄. Then Sym P contains rotations r_i around the lines passing through v_i and the center of the opposite face. These rotations take any face of P to any other.
 - (c) Show that Sym P acts transitively on the set of *flags*, i.e. triples (v, e, f), where v is a vertex of P, e is an edge, f is a face, and $v \in e \subset f$.

Let (v, e, f) and (v', e', f') be two flags. Using rotations r_i , we can take v to v'. Let $g_1v = v'$, $g_1 \in \text{Sym } P$, denote $e'' = g_1e$. If $v' = v_j$, then powers of rotation r_j act transitively on all three edges incident to v_j , so there is $g_2 \in \text{Sym } P$ such that $g_2(v') = v'$ and $g_2(e'') = e'$. So, g_2g_1 takes v to v' and e to e', denote $f'' = g_2g_1(f)$. If f'' = f' then we are done. Otherwise, observe that there are precisely two faces containing e', and they are taken to each other by a reflection in the plane passing through e' and the center of the opposite edge. Applying this reflection, we take f'' to f', preserving e' and v'.

- (d) (this is specific to cube!)
- (e) Compute the order of $\operatorname{Sym} P$.

The number of flags is $4 \cdot 3 \cdot 2 = 24$, the group Sym P acts on the set of flags with a single orbit and trivial stabilizer. Therefore, |Sym P| = 24.

(f) Show that $\operatorname{Sym} P$ is generated by reflections.

We have shown above that to take any flag to any flag we need rotations r_i and reflections. Now, every r_i is a product of two reflections, where the mirrors pass through the axis of r_i and any other vertex of P. There are six such reflections altogether (every mirror passes through an edge and the center of the opposite edge).

(g) Show that $\operatorname{Sym} P$ cannot be generated by two reflections.

If we assume that $\operatorname{Sym} P$ is generated by two reflections r_{α} and r_{β} , then the group would leave the space $\{\alpha, \beta\}^{\perp}$ of positive dimension invariant. However, $\operatorname{Sym} P$ is clearly irreducible.

Let (v, e, f) be a flag. Let $p_1 = v$, denote by p_2 the center of e, by p_3 the center of f, and by O the center of P (i.e., the origin of \mathbb{R}^3). Let C be the cone over triangle $p_1 p_2 p_3$ with apex O, i.e. the intersection of the three halfspaces: we take plane passing through $0p_i p_j$ and take the halfspace containing the third point p_k .

(h) Show that three reflections in the walls of C generate Sym P. Write down the relations among these generators (i.e., give a presentation of Sym P by generators and relations, where generators are the three reflections above).

Denote by s_i the reflection in plane $0p_jp_k$. Then it is easy to see that six reflections mentioned above are s_1 , s_2 , s_3 , $s_1s_2s_1$, $s_2s_3s_2$, and $s_1s_2s_3s_2s_1$.

We are left to find the relations, i.e. the orders of $s_i s_j$. Reflections s_1 and s_2 generate a dihedral group preserving f, so $(s_1 s_2)^3 = \text{id.}$ Similarly, reflections s_2 and s_3 generate a dihedral group preserving a neighboring face (which can be obtained from f by applying reflection $s_1 s_2 s_3 s_2 s_1$), so $(s_2 s_3)^3 = \text{id.}$ Finally, the reflections s_1 and s_3 commute (as the planes $0p_1p_2$ and $0p_2p_3$ are orthogonal), so

Sym
$$P = \langle s_1, s_2, s_3 \mid s_i^2, (s_1 s_2)^3, (s_2 s_3)^3, (s_1 s_3)^2 \rangle$$

Alternatively, one can use Theorem 7.7 from lectures.