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Topics in Combinatorics IV, Term 2

7 Reflection groups

7.1 Linear reflections and reflection groups

We consider $\mathbb{R}^n$ with standard Euclidean scalar product $(\cdot, \cdot)$ (i.e., dot product).

Definition 7.1. Let $\alpha \in \mathbb{R}^n$. A *reflection* with respect to α is a linear map $r_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$r_\alpha(v) = v - \frac{2(v, \alpha)}{(\alpha, \alpha)}\alpha$$

for $v \in \mathbb{R}^n$.

It is easy to see that r_α is characterized by the following two properties: it takes α to $-\alpha$ and preserves pointwise the orthogonal complement $\alpha^\perp$.

Exercise 7.2. $r_\alpha \in O_n(\mathbb{R})$, i.e. $(r_\alpha(u), r_\alpha(v)) = (u, v)$; also, r_α is an involution, i.e. $r_\alpha^2 = \mathrm{id}$.

Definition 7.3. A *reflection group* is a group generated by reflections.

Remark. Usually "reflection group" means a *discrete reflection group*, which requires some additional geometrical properties to hold (namely, the orbit of any point should not have limit points). We will mainly be interested in *finite* reflection groups, and for these there are no extra requirements.

Example 7.4.

- Consider vectors $\alpha = (1, 0)$ and $\beta = (\cos\frac{(m-1)\pi}{m}, \sin\frac{(m-1)\pi}{m})$. Then $r_\alpha r_\beta$ is a rotation by $2\pi/m$, and the group generated by r_α and r_β is a dihedral group of order $2m$ (denoted by $I_2(m)$).
- The symmetric group S_{n+1} acts on $\mathbb{R}^{n+1}$ by permutation of coordinates, preserving the hyperplane $V_0 = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum x_i = 0\}$. Every transposition (ij) is a reflection in a plane $x_i - x_j = 0$, i.e. with respect to the vector $e_i - e_j$. As S_{n+1} is generated by transpositions, it is a reflection group in $V_0 = \mathbb{R}^n$.

  One can also note that the action of S_{n+1} by permutation of coordinates of $\mathbb{R}^{n+1}$ preserves the affine hyperplane $V_1 = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum x_i = 1\}$, and it also preserves the positive orthant $C_+ = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_i \geq 0 \ \forall i \in [n+1]\}$. Thus, S_{n+1} preserves the *regular n-dimensional simplex* $V_1 \cap C_+$, acting on it by permutations of its vertices.

Definition 7.5. For reflection r_α, $\alpha \in \mathbb{R}^n$, the orthogonal complement $\alpha^\perp$ is called the *mirror* of r_α.

Lemma 7.6. *Let $g \in O_n(\mathbb{R})$, $\alpha \in \mathbb{R}^n$. Then $gr_\alpha g^{-1} = r_{g\alpha}$.*

Proof. We need to prove that $gr_\alpha g^{-1}$ fixes every point of $\langle g\alpha\rangle^\perp$ and takes $g\alpha$ to $-g\alpha$.

Let $(v, g\alpha) = 0$. Since $g \in O_n(\mathbb{R})$, this implies that $(g^{-1}v, \alpha) = 0$. Then

$$gr_\alpha g^{-1}(v) = g(r_\alpha(g^{-1}(v)) = g((g^{-1}(v)) = v,$$

so $gr_\alpha g^{-1}$ preserves $\langle g\alpha\rangle^\perp$ pointwise.

Also, $gr_\alpha g^{-1}(g\alpha) = gr_\alpha(\alpha) = g(-\alpha) = -g\alpha$, as required. $\square$

In general, what can we say about *finite* reflection groups in $\mathbb{R}^n$?

First, since every reflection is orthogonal, any reflection group is a subgroup of $O_n(\mathbb{R})$. Given a finite reflection group G in $\mathbb{R}^n$, Lemma 7.6 implies that the set of mirrors of reflections of G is invariant under the action of G (i.e., G permutes its mirrors).

The set of mirrors of G decomposes $\mathbb{R}^n$ into polyhedral cones – we call them *chambers*, and the mirrors bounding a chamber are called its *walls*.

Remark. Note that, due to the invariance of the set of mirrors under G, any two chambers sharing a wall are taken to each other by the reflection in the common wall. Indeed, if we take chambers C_1 and C_2 sharing a wall $\alpha^\perp$, we can consider $C' = r_\alpha C_1$. If C' is not a chamber, then there exists a mirror $\beta^\perp$ of reflection in G intersecting C'. Applying r_α to $\beta^\perp$, we see that the image intersects C_1, which contradicts C_1 being a chamber. Now, both C' and C_2 are chambers, and they clearly have a non-empty intersection, so they must coincide.

Recall that an action of a group on a set is *transitive* if the set consists of one orbit.

Theorem 7.7. *Let G be a finite reflection group in $\mathbb{R}^n$. Consider all mirrors of reflections of G, and take any connected component of the complement, call this chamber C_0. Denote $r_{\alpha_1}, \dots, r_{\alpha_k}$ the reflections with respect to the walls of C_0. Then*

(1) G is generated by $r_{\alpha_1}, \dots, r_{\alpha_k}$.

(2) G acts transitively on the set of chambers.

(3) The dihedral angles between walls of C_0 are of the type π/m_{ij}, $m_{ij} \in \mathbb{N}_{\geq 2}$.

(4) If $g \in G$ and $gC_0 = C_0$ then $g = \mathrm{id}$.

(5) G has presentation $G = \langle r_{\alpha_1}, \dots, r_{\alpha_k} \mid r_{\alpha_i}^2, (r_{\alpha_i} r_{\alpha_j})^{m_{ij}} \rangle$ (i.e. any relation on the generators follows from these two types of relations).

Proof. Denote r_{α_i} by s_i. Take any chamber C, connect it to C_0 by a path (which does not pass through an intersection of three or more chambers). Write down the sequence of chambers intersected by the path: $C_0, C_1, \dots, C_m = C$ (chambers may repeat in the sequence). Note that any two neighboring chambers in the sequence share a wall.

Since C_1 and C_0 share a wall (say, mirror of s_{i_1}), we can write $C_1 = s_{i_1} C_0$. By Lemma 7.6, walls of C_1 are precisely mirrors of reflections $s_{i_1} s_j s_{i_1}$, $j = 1, \dots, m$. Since C_2 and C_1 share a wall (say, mirror of $s_{i_1} s_{i_2} s_{i_1}$), we can write

$$C_2 = s_{i_1} s_{i_2} s_{i_1} C_1 = s_{i_1} s_{i_2} s_{i_1} (s_{i_1} C_0) = s_{i_1} s_{i_2} C_0.$$

Continuing along the path, we see that $C = C_m = s_{i_1} s_{i_2} \dots s_{i_m} C_0$, where s_{i_j} are reflections in the walls of C_0. This proves (2).

Moreover, we have proved that any reflection in G is conjugated to at least one of s_i, which proves (1).

Now, let us prove (3). Take any s_i and s_j and consider the group generated by them. If the angle is not $p\pi/m$, then the order of $s_i s_j$ is infinite, which contradicts finiteness of G. Further, if $p \neq 1$ (we may assume that p and m are coprime), then the walls are separated by another mirror (this is actually a question about dihedral groups), which implies that there exists a mirror of G intersecting the interior of C_0 in contradiction with its definition, so (3) is also proved.

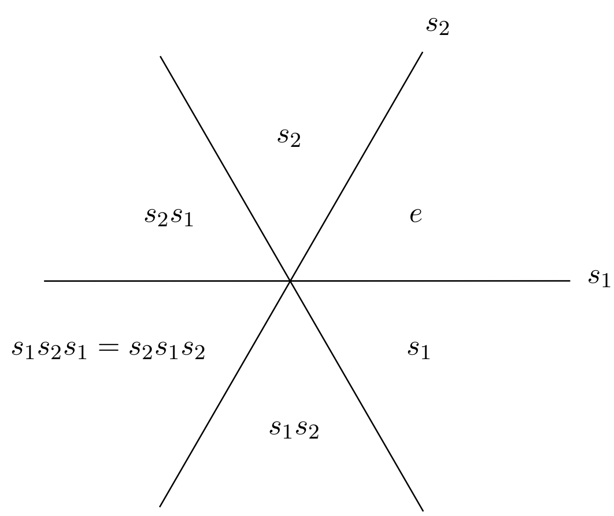
To prove (4) and (5) consider any word $s_{i_1} s_{i_2} \dots s_{i_m}$ realizing a path from C_0 to C_0 going through chambers $s_{i_1} s_{i_2} \dots s_{i_k} C_0$ for $k = 1, \dots, m-1$. Note that the relations in (5) do hold (as they hold in the corresponding dihedral groups). Moreover, they imply the same relations for reflections in walls of any chamber (due to Lemma 7.6). Further, using the relations we can contract the path to an empty one. More precisely, if a path intersects any single wall twice in a row, then applying the relation r^2 for r being the reflection in the corresponding wall we shorten the path; to go around an intersection of more than two chambers (which is a vector space of codimension two), one can use the relations (5) to substitute a subword of type $s_i s_j s_i \dots$ of length l by a word of type $s_j s_i s_j \dots$ of length $2m_{ij} - l$. Therefore, every path from C_0 to C_0 corresponds to a trivial element of G, and the word can be reduced to e by the required relations, which proves both (4) and (5).

□

Corollary 7.8. *Chambers of G are indexed by elements of G.*

Indeed, we can choose an initial chamber C_0 and then associate with any chamber $C = gC_0$ the corresponding element g.

Example. Consider $I_2(3) = S_3$, it has presentation $\langle s_1, s_2 \mid s_1^2, s_2^2, (s_1 s_2)^3 \rangle$.



Definition 7.9. Let a group G act on an open connected set X. An open connected domain $C \subset X$ is called a *fundamental domain* of the action if the following conditions are satisfied:

- $X = \bigcup\limits_{g \in G} \overline{gC}$, where $\overline{gC}$ denotes the closure of gC;
- for any nontrivial $g \in G$, $C \cap gC = \emptyset$;
- there are finitely many $g \in G$ such that $\overline{C} \cap \overline{gC} \neq \emptyset$.

Corollary 7.10. *Any chamber C of a finite reflection group G is a fundamental domain of the action of G on $\mathbb{R}^n$. In particular, chambers are also called* fundamental chambers.

7.2 Classification of finite reflection groups

Theorem 7.7(3) has the following elementary corollary.

Corollary 7.11. *Let C be a chamber of a finite reflection group, and let r_α and r_β be two generating reflections, where α and β are outward normals to walls of C. Then $(\alpha, \beta) \leq 0$. In other words, all angles of C are acute (or non-obtuse, depending on the agreement whether $\pi/2$ is acute or not).*

Definition 7.12. A system of vectors is *indecomposable* if it cannot be split into two subsets orthogonal to each other.

Lemma 7.13. *Let $\{e_i\}$ be a finite indecomposable system of vectors in $\mathbb{R}^n$ such that $(e_i, e_j) \leq 0$ for all $i \neq j$. Then either all e_i are linearly independent, or there exists a unique (up to scaling) linear dependence, and all its coefficients are positive.*

Proof. Assume $\{e_i\}$ are linearly dependent, and there is a linear dependence with some coefficients positive and some non-positive. Define index sets I and J so that coefficients of $e_i > 0$ are positive if $i \in I$ and non-positive if $j \in J$. We then can write

$$\sum_{i \in I} c_i e_i = \sum_{j \in J} c_j e_j,$$

where $c_i > 0$ and $c_j \geq 0$. Denote $\alpha = \sum_{i \in I} c_i e_i$ and $\beta = \sum_{j \in J} c_j e_j$. Then

$$(\alpha, \beta) = \sum_{i \in I, j \in J} c_i c_j (e_i, e_j).$$

Since $\alpha = \beta$, the value above is non-negative. At the same time, all c_i and c_j are non-negative, and all (e_i, e_j) are non-positive, so the product is non-positive. Therefore, we conclude that $(\alpha, \beta) = 0$, and thus $\alpha = \beta = 0$.

Take any $j \in J$, then $(\alpha, e_j) = (0, e_j) = 0$. At the same time, $0 = (\alpha, e_j) = (\sum_{i \in I} c_i e_i, e_j) = \sum_{i \in I} c_i (e_i, e_j)$. Since all $c_i > 0$, this implies that $(e_i, e_j) = 0$ for all $i \in I$. As this holds for every $j \in J$, we get a contradiction with indecomposability of $\{e_i\}$.

Now, assume that there are two positive linear dependencies $\sum c_i e_i = 0 = \sum a_i e_i$. Since all a_i and c_i are positive, we can scale them such that $a_1 = c_1$. If the dependencies are still distinct, then subtracting one dependence from another we get a new dependence with the coefficient before e_1 vanishing, which contradicts the statement we already proved.

□

Corollary 7.14. *If $\{e_i\}$ is a finite indecomposable system of vectors in $\mathbb{R}^n$ with $(e_i, e_j) \leq 0$ for $i \neq j$, then $\#\{e_i\} \leq n + 1$.*

Proof. Indeed, if there are $n + 2$ vectors, then there exists a linear dependence on any $n + 1$ of them, so there is a dependence with some coefficients vanishing, which contradicts Lemma 7.13. □

The next statement follows from the construction of the chambers.

Lemma 7.15. *Let C_0 be a chamber, and let α_i be outward normals to the walls of C_0. Then $C_0 = \{v \in \mathbb{R}^n \mid (v, \alpha_i) < 0\}$.*

Corollary 7.16. *In the assumptions of Lemma 7.15, $\#\{\alpha_i\} \leq n$. If G is* irreducible *(i.e., it has no invariant subspaces), then $\#\{\alpha_i\} = n$, and thus any chamber is a simplicial cone (i.e., any k walls intersect along an $(n - k)$-dimensional subspace).*

Proof. By Cor. 7.14, $\#\{\alpha_i\} \leq n + 1$. Assume there are $n + 1$ vectors, then there is a linear dependence $\sum c_i \alpha_i = 0$ with all $c_i > 0$. Thus, we can write

$$\alpha_{n+1} = -\sum_{i=1}^{n} \frac{c_i}{c_{n+1}} \alpha_i$$

Take any $v \in C_0$, we know that $(v, \alpha_i) < 0$ for every i. Then

$$(v, \alpha_{n+1}) = -(v, \sum_{i=1}^{n} \frac{c_i}{c_{n+1}} \alpha_i) = -\sum_{i=1}^{n} \frac{c_i}{c_{n+1}} (v, \alpha_i) > 0,$$

so we get a contradiction.

If $\#\{\alpha_i\} < n$, then the subspace $(\operatorname{span}\{\alpha_i\})^{\perp}$ has positive dimension (and thus is non-empty) and is clearly invariant (as every its point is fixed by every generating reflection, and thus by the whole group). □

In particular, we see that if G has no invariant subspaces, then the outward normals to every chamber compose a basis of $\mathbb{R}^n$.

Now, recall a simple fact from linear algebra. Let $\{e_i\}$ be a standard basis of $\mathbb{R}^n$, and let $\{\alpha_i\}$ be another basis. Denote by A the transformation matrix taking $\{e_i\}$ to $\{\alpha_i\}$ (i.e. columns of A are coordinates of $\{\alpha_i\}$ in terms of $\{e_i\}$). Then $(A^t A)_{ij} = (\alpha_i, \alpha_j)$. In particular, the matrix (α_i, α_j) (such matrix is called the *Gram matrix* of system of vectors $\{\alpha_i\}$) is the matrix of the dot-product in a different basis, and thus it is positive definite.

Therefore, to classify all finite reflection groups, we need (first) to find all positive-definite symmetric $n \times n$ matrices (a_{ij}) with $a_{ii} = 1$ and $a_{ij} = -\cos \frac{\pi}{m_{ij}}$ (we normalize outward normals to chambers to have unit length), and then see which of these matrices correspond to reflection groups.

Definition 7.17. Let G be a finite reflection group in $\mathbb{R}^n$, let $\{\alpha_i\}$ be outward unit normals to a chosen chamber. A *Coxeter diagram* of G is a (multi)graph defined as follows:

- vertices are indexed by α_i (or simply by natural numbers $1, \dots, n$);
- vertices i and j are joined by $m_{ij} - 2$ edges (or a simple edge with weight m_{ij}), where $(\alpha_i, \alpha_j) = -\cos \frac{\pi}{m_{ij}}$.

Example. Consider $I_2(3) = S_3 = \langle s_1, s_2 \mid s_1^2, s_2^2, (s_1 s_2)^3 \rangle$. The product of two generators has order 3, the only m_{ij} is $m_{12} = 3$. As $3 - 2 = 1$, the Coxeter diagram consists of two vertices joined by one simple edge.

In general, the Coxeter diagram of $I_2(m)$ consists of two vertices and $m - 2$ edges between them (or one edge with weight m written above the edge).

Remark 7.18. A Coxeter diagram of G is connected if and only if the collection of vectors $\{\alpha_i\}$ is indecomposable, and if and only if G is irreducible.

Lemma 7.19. *A Coxeter diagram of a finite irreducible reflection group G in $\mathbb{R}^n$ (where $n > 2$) does not contain*

(1) cycles;

(2) edges of multiplicity at least 4;

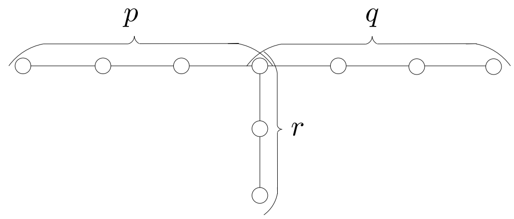
(3) two multiple edges;

(4) vertices of valence at least 4 (where by valence *we mean here the number of neighbors – this convention is not standard);*

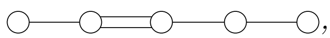
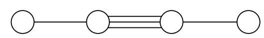
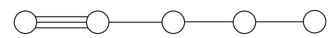
(5) both a vertex of valence 3 and a multiple edge;

(6) two vertices of valence at least 3;

(7) subdiagrams of type



with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1$

(8) subdiagrams ,  *and* 

Proof. Every Coxeter diagram corresponds to a positive definite symmetric matrix $A = (a_{ij}) = (\alpha_i, \alpha_j)$. The plan of the proof is the following: for every prohibited subdiagram we either find a principal submatrix of A with non-positive determinant, or find a linear combination of α_i which has non-positive square.

(1) Let vertices $1, \dots, k$ form a chordless cycle (i.e. this is a minimal cycle we can find with respect to inclusion), consider the vector $v = \alpha_1 + \dots + \alpha_k$. Then

$$v^2 = (v, v) = (\sum \alpha_i, \sum \alpha_i) = \sum (\alpha_i, \alpha_i) + 2 \sum_{i<j} (\alpha_i, \alpha_j) =$$

$$= k + 2 \sum_{i \leq k-1} (\alpha_i, \alpha_{i+1}) + 2(\alpha_k, \alpha_1) \leq k - 2k\frac{1}{2} = 0$$

as $(\alpha_i, \alpha_{i+1}) = -\|\alpha_i\| \|\alpha_{i+1}\| \cos \frac{\pi}{m_{i\,i+1}} = -\cos \frac{\pi}{m_{i\,i+1}} \leq -\frac{1}{2}$ for $m_{i\,i+1} \geq 3$.

(2) Suppose there is an edge of multiplicity at least 4 connecting vertices i and j. Since the diagram is connected and $n \geq 3$, there is a vertex k connected to one of i and j (say, to j). Since there are no cycles, k is not connected to i. Thus, we get a principal submatrix of A of the form

$$B(m_{ij}, m_{jk}) = \begin{pmatrix} 1 & -\cos \frac{\pi}{m_{ij}} & 0 \\ -\cos \frac{\pi}{m_{ij}} & 1 & -\cos \frac{\pi}{m_{jk}} \\ 0 & -\cos \frac{\pi}{m_{jk}} & 1 \end{pmatrix}.$$

Its determinant is equal to

$$\det B(m_{ij}, m_{jk}) = 1 - \cos^2 \frac{\pi}{m_{ij}} - \cos^2 \frac{\pi}{m_{jk}} \leq 1 - \left(\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2 = 1 - \frac{1}{4} - \frac{3}{4} = 0$$

as $\cos \frac{\pi}{m} \geq \frac{\sqrt{3}}{2}$ for $m \geq 6$.

(3) The determinant of matrix $B(4, 4)$ is equal to zero, and $\det B(m_{ij}, m_{jk})$ is a decreasing function of both arguments. Thus, two multiple edges cannot have a common vertex.

Suppose that there are two multiple edges in a Coxeter diagram, we may assume that they are connected by a sequence of simple edges. Assume the multiple edges connect vertices 1 with 2, and k with $k+1$, and there are simple edges between vertices i and $i+1$ for $i = 2, \dots, k-1$. Consider vector $v = \sum_{i=2}^{k} \alpha_i$. It is easy to see that $(v, v) = 1$, $(v, \alpha_1) = (\alpha_2, \alpha_1)$, and $(v, \alpha_{k+1}) = (\alpha_k, \alpha_{k+1})$. Then the Gram matrix of vectors $\alpha_1, v, \alpha_{k+1}$ is precisely $B(m_{12}, m_{k\,k+1})$, so it has a non-positive determinant.

(4) Let 1 be a vertex with at least four neighbors, choose any four neighbors, we may assume that the vertices have numbers $2, \dots, 5$. Then the Gram matrix of these five vectors has determinant zero if all edges are simple, and negative otherwise.

(5) If there is a vertex of valence 3 such that one of the edges incident to it is a double edge, then the determinant of the 4×4 submatrix is zero. If the multiplicity increases, the determinant becomes negative. If the multiple edge is not incident to a vertex of valence 3, proceed similarly to (3): take the sum of vectors "between" the multiple edge and the vertex of valence 3 and thus reduce the problem to the case above.

(6) Similar to (3): take the sum of vectors "between" two vertices of valence three, and then use (4).

(7) The determinant of the Gram matrix is equal to $pqr\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1\right)$.

(8) The determinants of the corresponding matrices are zero and negative respectively.

$\square$

As a result, we get the list of Coxeter diagrams shown in Table 7.1.

Table 7.1: **Coxeter diagrams.** Connected Coxeter diagrams of finite type

$\mathbf{A_n}$ $(n \geq 1)$

$\mathbf{B_n} = \mathbf{C_n}$

$\mathbf{D_n}$ $(n \geq 4)$

$\mathbf{F_4}$

$\mathbf{E_6}$

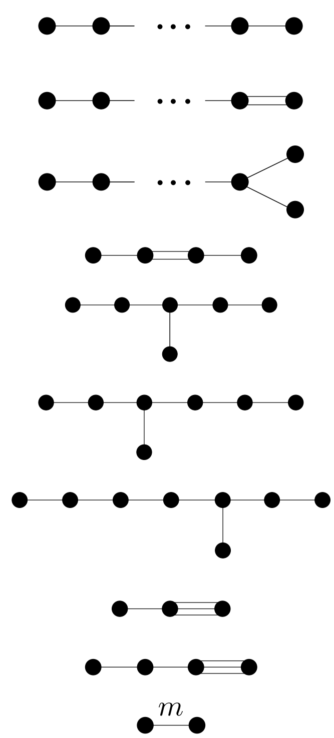
$\mathbf{E_7}$

$\mathbf{E_8}$

$\mathbf{H_3}$

$\mathbf{H_4}$

$\mathbf{I_2(m)}$



Remark. The diagrams without multiple edges (i.e., of types ADE) are called *simply-laced.*

Remark. For each diagram/matrix from Table 7.1 one can construct a system of vectors $\{\alpha_i\}$ with the corresponding Gram matrix. Indeed, as the matrix $\mathcal{A} = (\alpha_i, \alpha_j)_{i,j \leq n}$ is positive definite, there exists a basis in which the quadratic form is given by an identity matrix. Therefore, $\mathcal{A} = (A)^t A$ for some matrix A. The columns of A are precisely vectors α_i in the standard orthonormal basis.

Remark. Reflection groups with Coxeter diagram without vertices of valence 3 (i.e., A_n, B_n, F_4, H_3, H_4 and $I_2(m)$) are symmetry groups of *regular polytopes* in $\mathbb{R}^n$ (where a polytope is regular if its symmetry group acts transitively on flags, see HW 11 and 12).

8 Coxeter groups

8.1 Word metric

Coxeter groups can be understood as "abstract reflection groups".

Definition 8.1. Let $S = \{s_1, \dots, s_n\}$. A group with presentation $G = \langle S \mid s_i^2, (s_i s_j)^{m_{ij}} \rangle$, $m_{ij} = 2, \dots, \infty$, is called a *Coxeter group*. A pair (G, S) is called a *Coxeter system*. The number $n = |S|$ is the *rank* of (G, S).

It follows from Theorem 7.7 that every finite reflection group is a Coxeter group. If $m_{ij} = \infty$ then there is no relation between s_i and s_j.

Example. A group $G = \langle s_1, s_2 \mid s_i^2 \rangle$ is an *infinite dihedral group*. It can be understood as the group generated by reflections in two parallel lines in $\mathbb{R}^2$.

Definition 8.2. Let G be any group with a finite generating set S. Assume also that S is symmetric, i.e. for any $s \in S$ the inverse s^{-1} is also contained in S. A *word* is a finite sequence of elements of S. Any element $g \in G$ can be written as a word $s_{i_1} \dots s_{i_k}$, usually in a non-unique way. For given $g \in G$, the minimal possible k over all words representing g is called *the length* of g and is denoted by $l(g)$. A word $s_{i_1} \dots s_{i_k}$ is *reduced* if for $g = s_{i_1} \dots s_{i_k}$ we have $l(g) = k$. Two words w_1 and w_2 are *equivalent* if they represent the same element of G, notation $w_1 \sim_G w_2$.

Example. Consider $I_2(3) = S_3 = \langle s_1, s_2 \mid s_1^2, s_2^2, (s_1 s_2)^3 \rangle$. A word $s_1 s_2 s_1 s_2$ is not reduced as $s_1 s_2 s_1 s_2 = s_2 s_1$ in S_3, which is shorter. A word $s_2 s_1$ is reduced as the corresponding element cannot be written as a shorter product of generators.

Lemma 8.3. *Let $g, g' \in G$. Then*

(1) $l(gg') \leq l(g) + l(g')$;

(2) $l(g^{-1}) = l(g)$;

(3) $|l(g) - l(g')| \leq l(g' g^{-1})$.

Proof. (1) Let $s_1 \dots s_k$ be a reduced expression for g, and $s_{k+1} \dots s_{k+m}$ be a reduced expression for g'. Then $gg' = s_1 \dots s_{k+m}$, so $l(gg') \leq k + m = l(g) + l(g')$.

(2) Let $s_1 \dots s_k$ be a reduced expression for g. Then $g^{-1} = s_k^{-1} \dots s_1^{-1}$. Since S is symmetric, we have $l(g^{-1}) \leq l(g)$. Therefore, we have $l(g) = l((g^{-1})^{-1}) \leq l(g^{-1})$, and thus $l(g) = l(g^{-1})$.

(3) is left as an exercise (see HW 14).

$\square$

Exercise 8.4. Show that $d(g, g') = l(g' g^{-1})$ defines a metric on G.

8.2 Reflections, Exchange Condition and Deleting Condition

Definition 8.5. Let (G, S) be a Coxeter system. A *reflection* is an element of G conjugated to an element of S, the set of reflections is denoted by $R = \{gsg^{-1} \mid s \in S,\, g \in G\}$. Elements of S are called *simple reflections*.

If $w = s_1 \dots s_k$ is a word, an *R-sequence* $r(w)$ of w is defined as $r(w) = r_1, \dots, r_k$, where $r_i = (s_1 s_2 \dots s_{i-1}) s_i (s_{i-1} \dots s_1) \in R$.

Remark 8.6. For any $i \leq k$, $s_1 \dots s_i = r_i \dots r_1$.

Example. In $I_2(3)$, $r(s_1 s_2 s_1 s_2) = (s_1, s_1 s_2 s_1, s_1 s_2 s_1 s_2 s_1, s_1 s_2 s_1 s_2 s_1 s_2 s_1) = (s_1, s_1 s_2 s_1, s_2, s_1)$.

Remark. As we have seen while proving Theorem 7.7, in a finite reflection group a word w defines a path going through chambers. Then the R-sequence $r(w)$ consists of the reflections we need to apply to go from a chamber to the neighboring one.

Exercise 8.7. Show that $r(uv) = (r(u), ur(v)u^{-1})$.

Example. In $I_2(3)$, $r(s_1 s_2) = (s_1, s_1 s_2 s_1) = (r(s_1), s_1 r(s_2) s_1^{-1})$.

Definition 8.8. Given a word w and $r \in R$, define a non-negative integer $n(w, r)$ as the number of appearances of r in $r(w)$.

Theorem 8.9. *Let (G, S) be a Coxeter system, w, w_1, w_2 are words.*

(1) If $w_1 \sim_G w_2$, then $(-1)^{n(w_1, r)} = (-1)^{n(w_2, r)}$ for any $r \in R$.

(2) w is reduced if and only if $n(w, r) \leq 1$ for any $r \in R$.

Proof. By definition, two words w_1 and w_2 are equivalent if they can be taken to each other by a sequence of applications of relations. Every application of a relation v consists of removing or inserting a subword v in a given word, i.e. a word uw is substituted with uvw (or vice versa), where $v = s_i^2$ or $v = (s_i s_j)^{m_{ij}}$. Thus, to prove (1), we need to verify the statement for just one such move (for every relation).

By Exercise 8.7, $r(uw) = (r(u), ur(w)u^{-1})$, so

$$r(uvw) = (r(uv), uvr(w)v^{-1}u^{-1}) = (r(u), ur(v)u^{-1}, uvr(w)v^{-1}u^{-1})$$

Note that $uvr(w)v^{-1}u^{-1}$ coincides with $ur(w)u^{-1}$ as $v \sim_G e$ (and R-sequence is a sequence of group elements, not words). Comparing the sequences above, this implies that we just need to show that the sequence $ur(v)u^{-1}$ contains an even number of any reflection, or, equivalently, this holds for $r(v)$.

If $v = s_i^2$, then $r(v) = (s_i, s_i s_i s_i) = (s_i, s_i)$.

If $v = (s_i s_j)^{m_{ij}}$, then

$$r(v) = (s_i, s_i s_j s_i, \dots) = (s_i (s_j s_i)^k)_{k=0}^{2m_{ij}-1} = ((s_i (s_j s_i)^k)_{k=0}^{m_{ij}-1}, (s_i (s_j s_i)^k)_{k=m_{ij}}^{2m_{ij}-1}) =$$
$$= ((s_i (s_j s_i)^k)_{k=0}^{m_{ij}-1}, (s_i (s_j s_i)^k (s_j s_i)^{m_{ij}})_{k=0}^{m_{ij}-1})$$

Since $(s_j s_i)^{m_{ij}} = e$, we see that every entry in the R-sequence of v appears an even number of times, which completes the proof of (1).

To prove (2), assume first that all r_i in the R-sequence of w are distinct. If $s_1 \dots s_k$ is not reduced, then $s_1 \dots s_k \sim_G s_{i_1} \dots s_{i_m}$ for $m < k$. By (1), the R-sequence of $s_{i_1} \dots s_{i_m}$ must contain all k reflections from $r(w)$; however, as $m < k$, this is impossible.

Conversely, let $r_i = r_j$ for $i < j$, where $r(w) = (r_1, \dots, r_k)$. Recall that $r_i = (s_1 s_2 \dots s_{i-1}) s_i (s_{i-1} \dots s_1)$, and then

$$r_j = (s_1 s_2 \dots s_{j-1}) s_j (s_{j-1} \dots s_1) = (s_1 s_2 \dots s_{i-1})(s_i \dots s_{j-1}) s_j (s_{j-1} \dots s_i)(s_{i-1} \dots s_1)$$

Since $r_i = r_j$, we get $s_i \sim_G (s_i \dots s_{j-1}) s_j (s_{j-1} \dots s_i)$, which implies $s_i \dots s_j \sim_G s_{i+1} \dots s_{j-1}$, so we can shorten w, and thus w is not reduced.

□

Corollary 8.10. *Let $g \in G$, and let w be any word representing g. Then the number of distinct reflections in $r(w)$ is equal to $l(g)$.*

Remark 8.11. Theorem 8.9 implies that elements of R-sequence of a *reduced* word depend on a group element and not on a word representing it. Thus, for any $g \in G$ we can define the set $R(g)$ of all reflections appearing in the R-sequence of any reduced word representing g.

Further, as it was already pointed out, in case of finite reflection groups the R-sequence has a geometric meaning: given $g \in G$, elements of $R(g)$ are precisely all reflections whose mirrors separate the chamber gC from the initial chamber C.

Theorem 8.9 (and its proof) has several important consequences.

Corollary 8.12 (Deletion Condition)**.** *If $w = s_1 \dots s_k$ is not reduced, then $w \sim_G s_1 \dots \hat{s}_i \dots \hat{s}_j \dots s_k$ for some $i < j$ (where $\hat{s}$ means that the corresponding letter is removed from the word).*

Corollary 8.13 (Exchange Condition)**.** *Let $w = s_1 \dots s_k$ be reduced, $s \in S$, and sw is not reduced. Then $w \sim_G s s_1 \dots \hat{s}_i \dots s_k$ for some i.*

Proof. By the Deletion Condition, $sw \sim_G s s_1 \dots s_k$ with two letters removed. If one of them is s, then $sw \sim_G s_1 \dots \hat{s}_i \dots s_k$, so $w \sim_G s s_1 \dots \hat{s}_i \dots s_k$ as required.

Otherwise, $sw \sim_G s s_1 \dots \hat{s}_i \dots \hat{s}_j \dots s_k$. Then $w \sim_G s_1 \dots \hat{s}_i \dots \hat{s}_j \dots s_k$, which contradicts the fact w is reduced. □

The names "Deletion Condition" and "Exchange Condition" are explained by the following theorem (which we will not prove).

Theorem 8.14. *Let G be a group with a finite generating set S consisting of involutions (i.e. $s^2 = e$ for any $s \in S$). Then TFAE:*

(1) (G, S) is a Coxeter system;

(2) (G, S) satisfies the Deletion Condition;

(3) (G, S) satisfies the Exchange Condition.

8.3 Word problem

Let G be a group with a finite generating set S. Given a word w in the alphabet S, is there an algorithm to determine whether $w \sim_G e$?

In general, the answer to the question above is negative (P. S. Novikov, 1955, proved that there exist groups where the problem is undecidable). However, for Coxeter groups the answer is positive. We will show that any word can be transformed to a reduced word by a bounded number of some elementary operations. Then we show that any two reduced words representing the same element of the group are equivalent under these operations.

Definition 8.15. Let (G, S) a Coxeter system, w is a word. An *elementary M-operation* is either deleting a subword of the form ss or replacing a subword of the form $stst\dots$ of length m_{st} by the subword $tsts\dots$ of the same length (where m_{st} is the order of the element (st)). Clearly, any elementary M-operation preserves the element of the group. A word w is *M-reduced* if it cannot be shortened by a sequence of elementary M-operations. We call such a sequence *M-reduction*. Notice that any reduced word is obviously M-reduced.

We write $w \to w'$ (or $w' \leftarrow w$) if w' can be obtained from w by M-reduction. Notation $w \leftrightarrow w'$ means that $w \to w'$ and $w' \to w$ simultaneously.

Example. In S_3, take a word $s_1s_2s_1s_2s_1$. There are several M-reductions, one of them is shown below:

$$s_1s_2s_1s_2s_1 \to s_2s_1s_2s_2s_1 \to s_2s_1s_1 \to s_2$$

Recall that for any $g \in G$ the set $R(g)$ consists of elements of $r(w)$ for any reduced word w representing g (see Remark 8.11).

Lemma 8.16. *Let $s \in S$, and suppose that $s \in R(g)$. Then there exists a reduced word $w = sw'$ representing g.*

Proof. Take any reduced word $w = s_1 \dots s_k$ representing g, and let $r(w) = (r_1, \dots, r_k)$. Since $s \in R(g)$, we have $s = r_i$ for some $i \leq k$. Consider the word sw. By Exercise 8.7,

$$r(sw) = \{s, sr(w)s^{-1}\} = \{s, \dots, sr_{i-1}s, s, sr_{i+1}s, \dots\}$$

Therefore, by Theorem 8.9 sw is not reduced. By the Exchange Condition, g is represented by the word $ss_1 \dots \hat{s}_i \dots s_k$, as required. □

Lemma 8.17. *Let $w = st\dots$ be a reduced word representing $g \in G$, $s, t \in S$, and suppose that $s, t \in R(g)$. Then there exists reduced words representing g beginning with a subword $sts\dots$ and $tst\dots$ of length m_{st}.*

Proof. Let $w = vu$, where v is a maximal subword of the form $sts\dots$, and denote by q the length of v. Since w is reduced, $q \leq m_{st}$. We can assume that $q < m_{st}$, otherwise we have nothing to prove. We can write

$$r(w) = r(vu) = (r(v), vr(u)v^{-1}) = (s, sts, \dots, (st)^{q-1}s, vr(u)v^{-1})$$

Notice that $t \notin r(v)$: if $t = (st)^{p-1}s$ then $(st)^p = 1$ for some $p \leq q < m_{st}$, so we would get a contradiction. Since $t \in r(w)$ by the assumption, this implies that $t \in vr(u)v^{-1}$. Since tw is not reduced (see Theorem 8.9), by the Deleting Condition $w \sim_G tvu'$, where u' is obtained from u by omitting one letter. Now, tvu' starts with $tv = tsts\dots$ of length $q+1$. Applying induction, we obtain the statement of the lemma. □

Denote by $lt(w)$ the length of a word in (G, S).

Exercise 8.18. (1) If $w \sim_G w'$, then $lt(w) - lt(w') \equiv 0 \bmod 2$.

(2) Let $r \in R$ and $g \in G$. Show that $r \in R(g)$ if and only if $l(rg) < l(g)$.

We can now formulate the main theorem of this section.

Theorem 8.19 (Matsumoto's Theorem)**.** *Let w_1 and w_2 be M-reduced words, and $w_1 \sim_G w_2$. Then $lt(w_1) = lt(w_2)$, and $w_1 \leftrightarrow w_2$. In particular, any M-reduced word is reduced.*

Proof. The proof is by induction on the maximum of $(lt(w_1), lt(w_2))$. We can assume that $lt(w_1) \geq lt(w_2)$.

Case 1: $lt(w_1) > lt(w_2)$.
Let $s \in S$ be the first letter of w_1, i.e. $w_1 = sw_1'$ and w_1' is M-reduced. Then $w_1' \sim_G sw_2$. Denote by w_2' an M-reduced word such that $sw_2 \to w_2'$. By Exercise 8.18, we have $lt(w_1) \geq lt(w_2) + 2$. Thus, we see that $lt(w_1') \geq lt(sw_2) \geq lt(w_2')$, and $w_1' \sim_G w_2'$. By the induction assumption, $lt(w_1') = lt(w_2')$ and $w_1' \leftrightarrow w_2'$. Therefore, $lt(w_1') = lt(sw_2) = lt(w_2')$, so sw_2 is M-reduced and $sw_2 \leftrightarrow w_1'$. Thus, $w_2 \leftarrow ssw_2 \leftrightarrow sw_1' = w_1$, that contradicts the fact that w_1 is M-reduced.

Case 2: $lt(w_1) = lt(w_2)$, and either w_1 or w_2 is not reduced.
We assume that w_2 is not reduced. Then there exists a reduced word $w_2' \sim_G w_2$, $lt(w_2') < lt(w_2)$. Now we have a pair of M-reduced words $w_2' \sim_G w_2$ with $lt(w_2') < lt(w_2)$, which is impossible by Case 1.

Case 3: w_1 and w_2 are reduced, and $w_1 = sw_1'$, $w_2 = sw_2'$ for some $s \in S$.
In this case we see that $w_1' \leftrightarrow w_2'$ by the induction assumption, so $w_1 \leftrightarrow w_2$.

Case 4: w_1 and w_2 are reduced, and $w_1 = s_1 w_1'$, $w_2 = s_2 w_2'$, $s_1 \neq s_2$.
Let $g \in G$ be the element represented by w_1 and w_2. Then $s_1, s_2 \in R(g)$. By Lemma 8.17, there exists $w_3 \sim_G w_1, w_2$, $w_3 = vu$, where $v = s_1 s_2 \dots$ has length $m_{s_1 s_2}$. Denote by v' the word $v' = s_2 s_1 \dots$ of length $m_{s_1 s_2}$, let $w_4 = v'u$. By Case 3, $w_1 \leftrightarrow w_3 \leftrightarrow w_4 \leftrightarrow w_2$, which completes the proof.

□

Corollary 8.20. *w represents $e \in G$ if and only if $w \to \emptyset$.*

As the number of M-reductions is finite, this provides an algorithm.

8.4 Coxeter groups and reflection groups

We have already seen that finite reflection groups are Coxeter groups.

Conversely, let (G, S) be a Coxeter system. We construct a discrete linear group in real vector space of dimension $|S|$, such that for any two elements $s, t \in S$ the order of (st) is equal to m_{st}, and $s^2 = e$.

Let V be a real vector space with basis $\{e_s,\ s \in S\}$. Define a symmetric bilinear form on V by

$$(e_s, e_t) = -\cos(\pi/m_{st})$$

In particular, $(e_s, e_s) = 1$, and $(e_s, e_t) = -1$ if $m_{st} = \infty$.

Exercise 8.21. Show that $\forall s \in S \;\; V = \langle e_s \rangle \oplus \langle e_s \rangle^{\perp}$.

Denote by H_s the hyperplane $\langle e_s \rangle^{\perp}$. A *reflection* in H_s is given by

$$r_s(x) = x - 2(x, e_s)e_s$$

Now let $s, t \in S$, $s \neq t$.

If $m_{st} \neq \infty$, the bilinear form is positive definite on the plane $\langle e_s, e_t \rangle$. Thus, $V = \langle e_s, e_t \rangle \oplus \langle e_s, e_t \rangle^{\perp}$. Reflections r_s and r_t fix $\langle e_s, e_t \rangle^{\perp}$, and generate a dihedral group on $\langle e_s, e_t \rangle$. Therefore, the order of (st) equals m_{st}.

If $m_{st} = \infty$, $r_s(e_t) = e_t + 2e_s$ and $r_t(e_s) = e_s + 2e_t$. Thus, the restriction of r_s, r_t and $r_s r_t$ onto the plane $\langle e_s, e_t \rangle$ is given by matrices

$$r_s \sim \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \qquad r_t \sim \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \qquad r_s r_t \sim \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$$

But the latter matrix has determinant 1 and trace 2 (and is not the identity matrix), so it is conjugated to the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, which has infinite order. Therefore, the order of (st) is infinite.

Definition 8.22. Let (G, S) be a Coxeter system. A *Coxeter diagram* of G is a (multi)graph defined as follows:

- vertices are indexed by s_i (or simply by natural numbers $1, \ldots, n$);
- vertices i and j are joined by $m_{ij} - 2$ edges (or a simple edge with weight m_{ij}), where $(s_i s_j)^{m_{ij}} = e$, if $m_{ij} \in \mathbb{N}$, and by a thick edge if $m_{ij} = \infty$.

Remark. The definition agrees with the definition of the Coxeter diagram of a finite reflection group.

Example. A Coxeter diagram of an infinite dihedral group consists of one thick edge.

Theorem 8.23. *Let (G, S) be a Coxeter system. Then G is finite if and only if the bilinear form on $\mathbb{R}^{|S|}$ defined above is positive definite.*

Proof. If the form is positive definite, then the Coxeter diagram coincides with one of the Coxeter diagrams of finite reflection groups. In particular, G has the same presentation as one of the finite reflection groups, so it is finite.

Conversely, if the form is not positive definite, then the Coxeter diagram contains one of "prohibited" subdiagrams from Lemma 7.19 (or a thick edge which corresponds to a pair of simple reflections such that its product has an infinite order). According to HW 15.2, the corresponding subgroup is also a Coxeter group with the presentation defined by the subdiagram. One now needs to find an element of infinite order in each of the corresponding Coxeter groups, which we leave as an exercise. $\square$

We will see a more conceptual proof of the theorem later on.

Example. Consider a subdiagram consisting of two multiple edges, i.e. a group with presentation $\langle s_1, s_2, s_3 \mid s_i^2, (s_1s_2)^k, (s_2s_3)^m, (s_1s_3)^2 \rangle$ with $k, m \geq 4$. Let $w = s_1s_2s_3$. Then every word w^k is M-reduced: the only M-operation applicable to it is interchanging s_1s_3 with s_3s_1, which does not change the length. In particular, the corresponding group element is of infinite order.

9 Root systems

9.1 Definitions

Definition 9.1. A set $\Delta \subset \mathbb{R}^n$ is a *root system* of rank n if the following conditions hold:

(1) Δ is finite and spans $\mathbb{R}^n$;

(2) the expression $\langle \alpha \mid \beta \rangle \stackrel{\text{def}}{=} \frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$ for all $\alpha, \beta \in \Delta$;

(3) $r_\alpha(\Delta) = \Delta$ for any $\alpha \in \Delta$.

A root system is *reduced* if the following holds:

(4) if $\alpha \in \Delta$ and $c\alpha \in \Delta$ for some $c \in \mathbb{R}$ then $c = \pm 1$.

By default, we will assume that all root systems are reduced.

Example 9.2. Take a reflection group of type A_2 and normalize *inward* normals α, β to the walls of a chamber to have length $\sqrt{2}$. Then $(\alpha, \beta) = -1$, so $\langle \alpha \mid \beta \rangle = -1$. Now, a normal to the third mirror is $\alpha + \beta$, $\|\alpha + \beta\|^2 = (\alpha + \beta, \alpha + \beta) = 2$ as well. In particular, $\langle \alpha + \beta \mid \alpha \rangle = \langle \alpha \mid \alpha \rangle + \langle \beta \mid \alpha \rangle = 1$. So, we see that the collection of six vectors $\pm\alpha, \pm\beta, \pm(\alpha + \beta)$ is a root system.

Lemma 9.3. *Let $\angle(\alpha, \beta) = \varphi$. Then $\langle \alpha \mid \beta \rangle \langle \beta \mid \alpha \rangle = 4\cos^2 \varphi$. In particular, $\varphi \in \{\frac{k\pi}{2}, \frac{k\pi}{3}, \frac{k\pi}{4}, \frac{k\pi}{6}\}$.*

This can be proved by an elementary computation.

Definition 9.4. Given $\alpha \in \Delta$, denote $H_\alpha = \alpha^\perp$ the mirror of the reflection r_α. A connected component of $\mathbb{R}^n \setminus \bigcup\limits_{\alpha \in \Delta} H_\alpha$ is called a *Weyl chamber* of Δ.

Given a Weyl chamber C, denote by $\Pi = \Pi(C)$ the set of inward normals to the walls of C lying in Δ. Π is called a *set of simple roots* of Δ (defined by C). Then $C = \{v \in \mathbb{R}^n \mid (\alpha, v) > 0 \,\forall\, \alpha \in \Pi\}$.

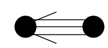
The group $W = W(\Delta)$ generated by all reflections r_α, $\alpha \in \Delta$, is called the *Weyl group* of Δ. If we denote $S = \{r_\alpha \mid \alpha \in \Pi\}$, then the pair (W, S) is a Coxeter system. All sets of simple roots are equivalent under action of W.

Given Π, the matrix $A = (a_{ij}) = (\langle \alpha_i \mid \alpha_j \rangle)$ is called a *Cartan matrix* of Δ.

Example 9.5. Weyl group of type G_2 is generated by two reflections in lines forming an angle $\pi/6$. Since the minimal angle between mirrors is $\varphi = \pi/6$, we have $4\cos^2\varphi = 3$, and thus $\langle \alpha \mid \beta \rangle \langle \beta \mid \alpha \rangle = 3$, where α, β simple roots. This implies that $\langle \alpha \mid \beta \rangle = -1$ and $\langle \beta \mid \alpha \rangle = -3$ (or vice versa). This, in its turn, implies that $\frac{(\beta,\beta)}{(\alpha,\alpha)} = 3$.

Therefore, we may take the root system consisting of six vectors $e^{\frac{k\pi i}{3}}$ of unit length and of six vectors $\sqrt{3}e^{\frac{\pi i}{6} + \frac{k\pi i}{3}}$ of length $\sqrt{3}$. The Cartan matrix is $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$.

Definition 9.6. *Dynkin diagram* of a root system Δ is a graph with oriented (multiple) edges. Vertices are indexed by elements of Π; if $|a_{ij}| \geq |a_{ji}|$, then there are $|a_{ij}|$ edges between i and j; the edges are oriented from i to j if $a_{ij} \neq a_{ji}$, otherwise they are non-oriented.

Example 9.7. The Dynkin diagram for the root system of type G_2 is 

9.2 Classification

Lemma 9.8. *(a) A set of simple roots Π defines Δ uniquely.*

(b) Let G be a finite reflection group, let $\alpha_1, \ldots, \alpha_n$ be inward normals to the walls of a chamber C of G, and assume that $\langle \alpha_i \mid \alpha_j \rangle \in \mathbb{Z}$ for all i, j. Then there exists a unique root system Δ with $W(\Delta) = G$ and $\Pi = \{\alpha_i\}$.

Proof. To prove (a) we just need to observe that $\Delta = W\Pi$.

To prove (b), consider a *lattice* $L = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_n = \{c_1\alpha_1 + \cdots + c_n\alpha_n \mid a_i \in \mathbb{Z}\}$. Since $r_{\alpha_j}(\alpha_i) = \alpha_i - \langle \alpha_i \mid \alpha_j \rangle \alpha_j$ and $\langle \alpha_i \mid \alpha_j \rangle \in \mathbb{Z}$, L is invariant with respect to all r_{α_j}, and thus is invariant with respect to G.

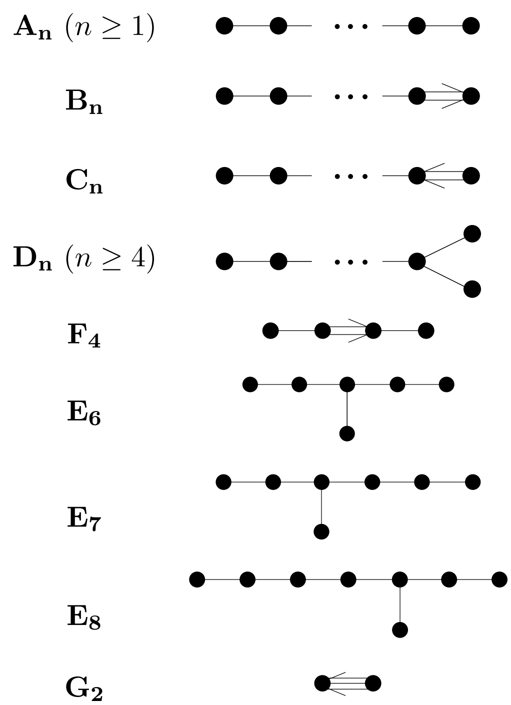
Notice also that the set $G\{\alpha_i\}$ contains normals to all mirrors of G. Define Δ to be the set of all *primitive* vectors of L orthogonal to mirrors of G (a vector v is primitive in L if $v \neq cv'$ for $v' \in L$ and $|c| > 1$), i.e.

$$\Delta = \{\alpha \in L \mid r_\alpha \in G,\ \alpha \neq cv \ \forall v \in L, c \in \mathbb{Z}_{>1}\}$$

Then Δ is finite, invariant with respect to $W = G$, reduced, and for every $\alpha, \beta \in \Delta$ we have $r_\alpha(\beta) = \beta - \langle \beta \mid \alpha \rangle \alpha \in L$, which implies $\langle \beta \mid \alpha \rangle \alpha \in L$, and thus $\langle \beta \mid \alpha \rangle \in \mathbb{Z}$.

∎

Table 9.1: **Dynkin diagrams.** Dynkin diagrams of finite irreducible root systems



Corollary 9.9 (Classification of irreducible root systems)**.** *A root system is called* irreducible *if its Dynkin diagram is connected (or, equivalently, if its Weyl group is irreducible). Table 9.1 contains Dynkin diagrams of all irreducible root systems.*

Remark. Similarly to finite reflection groups, Dynkin diagram without multiple edges (and the corresponding root systems) are called *simply-laced.* It can be seen from the classification of root systems that non-simply-laced root systems contain roots of precisely two lengths, these are called *short roots* and *long roots*. For root systems of types B_n, C_n and F_4, the ratio of lengths of long and short roots is $\sqrt{2}$, while for G_2 it is $\sqrt{3}$.

9.3 Structure of finite root systems

Definition 9.10. Let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ be a set of simple roots, and let $\alpha = \sum c_i\alpha_i \in \Delta$. A root α is *positive* if all $c_i \geq 0$, and *negative* if all $c_i \leq 0$. The set of positive roots is denoted by Δ^+, and the set of negative roots is denoted by Δ^-.

Example 9.11. All simple roots are positive.

Theorem 9.12. $\Delta = \Delta^+ \cup \Delta^-$.

Proof. Let C be the Weyl chamber such that $\Pi = \Pi(C)$. Take any $\alpha \in \Delta$, then either $(\alpha, v) > 0$ for any $v \in C$, or $(\alpha, v) < 0$ for any $v \in C$.

The theorem follows from the following lemma.

Lemma 9.13. *Let $v \in C$, $\alpha \in \Delta$. If $(\alpha, v) > 0$ then $\alpha \in \Delta^+$, and if $(\alpha, v) < 0$ then $\alpha \in \Delta^-$.*

Proof of Lemma 9.13. Define the dual basis $\{\omega_i\}$ to $\{\alpha_i\}$ by $(\alpha_i, \omega_j) = 1$ if $i = j$ and $(\alpha_i, \omega_j) = 0$ otherwise. Then ω_i is orthogonal to all but one simple roots and $(\omega_i, \alpha_i) = 1 > 0$, so ω_i is a one-dimensional face of C. Thus, C can be written as $C = \{v \in \mathbb{R}^n \mid v = a_1\omega_1 + \cdots + a_n\omega_n,\ a_i > 0\}$. In particular, given $u \in \mathbb{R}^n$, we have $(u, v) < 0$ for any $v \in C$ if and only if $(u, \omega_i) \leq 0$ for all i (and similar for positive scalar product).

Now, let $\alpha = c_1\alpha_1 + \cdots + c_n\alpha_n \in \Delta$, $v \in C$. Since (α, u) has the same sign for all $u \in C$, we have $(\alpha, v) < 0$ if and only $(\alpha, \omega_i) \leq 0$ for all i. But $(\alpha, \omega_i) = c_i$, so $(\alpha, v) < 0$ if and only if $c_i \leq 0$ for all i, i.e. $\alpha \in \Delta^-$. The other case is identical.

□

The lemma shows that $\Delta^+ = \{\alpha \in \Delta \mid (\alpha, v) > 0\ \forall v \in C\}$ and $\Delta^- = \{\alpha \in \Delta \mid (\alpha, v) < 0\ \forall v \in C\}$, which proves the theorem.

□

Remark 9.14. The definition of positivity clearly depends on the chamber. Theorem 9.12 shows that positivity can also be defined by (almost) any element f_v of the dual vector space, where $f_v(\alpha) = (v, \alpha)$: α is positive if and only if $f_v(a) > 0$ – this is equivalent to α being positive with respect to the chamber containing v.

Exercise 9.15. Let $\alpha, \beta \in \Delta$.

(a) If $(\alpha, \beta) > 0$ then either $\alpha - \beta \in \Delta$ or $\alpha = \beta$.

(b) If $(\alpha, \beta) < 0$ then either $\alpha + \beta \in \Delta$ or $\alpha = -\beta$.

9.4 Root poset and the highest root

Definition 9.16. Let $\alpha, \beta \in \Delta^+$. We say that $\alpha \geq \beta$ if $\alpha - \beta = \sum c_i\alpha_i$ with $\alpha_i \in \Pi$, all $c_i \geq 0$. This supplies Δ^+ with a partial order, and thus defines the *root poset.*

Exercise. Draw the Hasse diagram of the root poset of type A_2.

Lemma 9.17. *Let C be a Weyl chamber of Δ, $\Pi = \{\alpha_i\}$ be simple roots, and $\overline{C}$ be the closure of C. There exists a unique maximal element $\widetilde{\alpha}_0$ of the root poset, i.e. $\widetilde{\alpha}_0 \geq \alpha$ for every $\alpha \in \Delta^+$ (it is called the* highest root*). Moreover, $\widetilde{\alpha}_0 \in \overline{C}$, and $\widetilde{\alpha}_0 = \sum\limits_{\alpha_i \in \Pi} c_i\alpha_i$ with all $c_i > 0$.*

Proof. First, let α be a maximal element of the root poset. If $(\alpha, \alpha_i) < 0$ for some i, then by Exercise 9.15 $\alpha + \alpha_i \in \Delta$ (or $\alpha = -\alpha_i$ which cannot happen as $\alpha \in \Delta^+$), which contradicts maximality. Therefore, $(\alpha, \alpha_i) \geq 0$ for all i, and thus $\alpha \in \overline{C}$.

Further, write $\alpha = \sum\limits_{\alpha_i \in \Pi} c_i\alpha_i$, $c_i \geq 0$. Denote $I = \{i \mid c_i > 0\}$ and $J = \{j \mid c_j = 0\}$, and assume $J \neq \emptyset$. As the Dynkin diagram is connected, there exist neighboring nodes $i \in I$ and $j \in J$, so $(\alpha_i, \alpha_j) \neq 0$. Then $(\alpha, \alpha_j) < 0$, which is not the case as we have proved above. Therefore, J is empty, and thus all coefficients of α are positive.

Finally, assume there are two maximal elements in the root poset, α and β. Then $(\alpha, \beta) = \sum c_i(\alpha_i, \beta)$. As we have proved, for every i we have $(\alpha_i, \beta) > 0$ and $c_i > 0$, so $(\alpha, \beta) > 0$. By Exercise 9.15, either $\alpha = \beta$ (and we are done) or $\alpha - \beta \in \Delta$. In the latter case, either $\alpha - \beta \in \Delta^+$ or $\beta - \alpha \in \Delta^+$, so either $\alpha > \beta$ or $\beta > \alpha$, and thus we come to a contradiction.

□

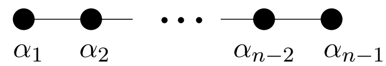
Remark 9.18. The highest root is always long (see HW 16.2).

10 Construction of finite root systems

Let $e_1, \dots, e_n$ be an orthonormal basis of $\mathbb{R}^n$.

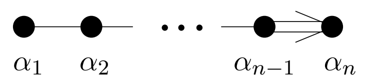
10.1 Classical series: ABCD

10.1.1 A_{n-1}



Consider the hyperplane $H = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum x_i = 0\} \simeq \mathbb{R}^{n-1}$. Define $\Delta = \{e_i - e_j\}$, $i \neq j$. Then $r_{e_i - e_j}$ preserves H and acts by permutation of i-th and j-th coordinates. $W \simeq S_n$ (and thus $|W| = n!$), simple roots are $\Pi = \{\alpha_i = e_i - e_{i+1}\}$. The corresponding Weyl chamber is given by inequalities $C = \{x \in H \mid x_1 > x_2 > \dots > x_n\}$, positive roots are $\{e_i - e_j\}$ for $i < j$. The highest root is $\sum\limits_{\alpha_i \in \Pi} \alpha_i = e_1 - e_n$.

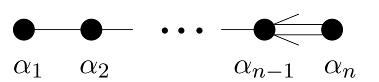
10.1.2 B_n



Define $\Delta = \{\pm e_i \pm e_j, \pm e_i\}$, $i < j$. Then $r_{e_i - e_j}$ acts by permutation of i-th and j-th coordinates, and r_{e_i} acts by change of sign of i-th coordinate. Simple roots are $\Pi = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \dots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_n\}$. The corresponding Weyl chamber is given by inequalities $C = \{x \in \mathbb{R}^n \mid x_1 > x_2 > \dots > x_n > 0\}$, positive roots are $\{e_i \pm e_j\}$ for $i < j$ and $\{e_i\}$. The highest root is $\alpha_1 + \sum\limits_{i=2}^{n} 2\alpha_i = e_1 + e_2$.

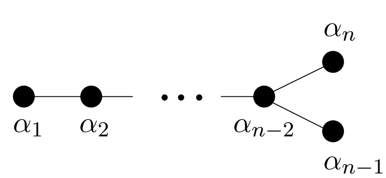
To determine the order of the Weyl group W, observe that if an element of W fixes the basis $\{e_i\}$ then it is identity, and thus we just need to count the bases W sends $\{e_i\}$ to. Every reflection in W is either a permutation of the basis vectors, or a change of sign of a basis vector, or a combination of these ($r_{e_i + e_j}$ takes e_i to $-e_j$ and e_j to $-e_i$ leaving all others intact). Then any element of W is also a permutation of the basis vectors and a change of signs of some basis vectors. Applying reflections in long roots, we can get any permutation, and applying reflections in short roots we can get any sign change. Therefore, there are $n!2^n$ configurations we can get, so $|W| = n!2^n$.

10.1.3 C_n



Root system C_n differs from B_n by the lengths of roots only. We can take $\Delta = \{\pm e_i \pm e_j, \pm 2e_i\}$, $i < j$. Simple roots are $\Pi = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \dots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = 2e_n\}$, positive roots are $\{e_i \pm e_j\}$ for $i < j$ and $\{2e_i\}$. The highest root is $\sum\limits_{i=1}^{n-1} 2\alpha_i + \alpha_n = 2e_1$. The Weyl group (and Weyl chamber) coincides with the one for B_n.

10.1.4 D_n



Define $\Delta = \{\pm e_i \pm e_j\}$, $i < j$ – this is precisely the set of long roots of B_n. Simple roots are $\Pi = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \ldots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_{n-1} + e_n\}$. The corresponding Weyl chamber is given by inequalities $C = \{x \in \mathbb{R}^n \mid x_1 > x_2 > \cdots > x_n, x_{n_1} + x_n > 0\}$, positive roots are $\{e_i \pm e_j\}$ for $i < j$. The highest root is $\alpha_1 + \sum\limits_{i=2}^{n-2} 2\alpha_i + \alpha_{n-1} + \alpha_n = e_1 + e_2$.

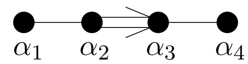
The order of the Weyl group W can be determined similarly to the B_n case: every reflection is either a transposition of two basis vectors, or a transposition composed with the change of both signs. Thus, we can get any permutation, but the number of changes of sign should be even. This implies that there are $n!2^{n-1}$ configurations we can get, so $|W| = n!2^{n-1}$.

Another way to find the order of the group is to look at the Weyl chamber: it is a union of two copies of the Weyl chamber of B_n (reflected in the hyperplane $x_n = 0$). Therefore, $[W(B_n) : W(D_n)] = 2$, and the result follows.

Section 10.2 is NON-EXAMINABLE.

10.2 Exceptional root systems: EFG

10.2.1 F_4



Consider the lattice $L \subset \mathbb{R}^4$ defined as follows: $L = \mathrm{span}_{\mathbb{Z}}\{e_1, e_2, e_3, e_4, \frac{1}{2}(e_1+e_2+e_3+e_4)\}$. In other words, L consists of all linear combinations of five vectors above with integer coefficients. Define

$$\Delta = \{v \in L \mid (v,v) = 1 \text{ or } (v,v) = 2\} = \{\pm e_i, \pm e_i \pm e_j, \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4), i < j\}$$

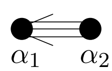
The reflections in vectors of Δ preserve L, and thus the whole group W preserves L, which implies W preserves Δ. Simple roots are $\Pi = \{\alpha_1 = e_2 - e_3, \alpha_2 = e_3 - e_4, \alpha_3 = e_4, \alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)\}$. The corresponding Weyl chamber is given by inequalities $C = \{x \in \mathbb{R}^4 \mid x_2 > x_3 > x_4 > 0, x_1 > x_2 + x_3 + x_4\}$, positive roots are $\{e_i \pm e_j\}$ for $i < j$, $\{e_i\}$, $\{\frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)\}$. The highest root is $2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 = e_1 + e_2$.

To find the order of the Weyl group observe that W acts transitively on long roots (cf. HW 13.1). There are 24 long roots, so we are left to find the order of the stabilizer of one of them. Consider the highest root $\widetilde{\alpha}_0 = e_1 + e_2$, it is orthogonal to all simple roots except for α_1. Due to HW 16.4, its stabilizer is generated by all reflections with respect to simple roots lying in $\widetilde{\alpha}_0^{\perp}$, which in our case generate Weyl group C_3. Thus, $|W| = 24|W(B_3)| = 24 \cdot 48 = 1152$.

Remark. $W(F_4)$ is a symmetry group of a regular 4-dimensional polytope, a 24-*cell*.

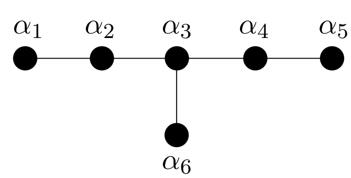
Remark. $W(F_4)$ contains $W(B_4)$ as a subgroup of index 3.

10.2.2 G_2



Here the Weyl group is a dihedral group of type $I_2(6)$, we can associate $\mathbb{R}^2$ with $\mathbb{C}$. Then $\Delta = \{\exp(\frac{k\pi i}{3}), \sqrt{3}\exp(\frac{\pi i}{6} + \frac{k\pi i}{3}), k = 0, \dots 5\}$. The simple roots are $\alpha_1 = 1, \alpha_2 = \sqrt{3}\exp(\frac{5\pi i}{6})$, the positive roots are all with argument in $[0, \pi)$. The highest root is $3\alpha_1 + 2\alpha_2 = \sqrt{3}i$.

10.2.3 E_6



Consider $\mathbb{R}^7$ with orthonormal basis $\{e_1, \dots, e_6, e\}$. Let $H = \{(x_1, \dots, x_6, y) \in \mathbb{R}^7 \mid \sum x_i = 0\} \simeq \mathbb{R}^6$. Define

$$L = \mathrm{span}_{\mathbb{Z}}\{e_1, e_2, e_3, e_4, e_5, e_6, \sqrt{2}e, \frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + \sqrt{2}e)\} \cap H,$$

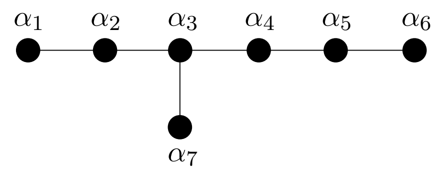
and let

$$\Delta = \{v \in L \mid (v,v) = 2\} = \{e_i - e_j, \pm\sqrt{2}e, \frac{1}{2}(e_i + e_j + e_k - e_p - e_q - e_r \pm \sqrt{2}e)\},$$

where all indices are distinct. Simple roots are $\Pi = \{\alpha_1 = e_1 - e_2, \ldots, \alpha_5 = e_5 - e_6, \alpha_6 = \frac{1}{2}(-e_1 - e_2 - e_3 + e_4 + e_5 + e_6 + \sqrt{2}e)\}$. The corresponding Weyl chamber is given by inequalities $C = \{(x, y) \in H \mid x_1 > x_2 > x_3 > x_4 > x_5 > x_6, \sqrt{2}y + x_4 + x_5 + x_6 > x_1 + x_2 + x_3\}$, positive roots are $\{e_i - e_j\}$ for $i < j$, $\sqrt{2}e$, $\{\frac{1}{2}(e_i + e_j + e_k - e_p - e_q - e_r + \sqrt{2}e)\}$. The highest root is $\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6 = \sqrt{2}e$.

There are 72 roots, the highest root is orthogonal to simple roots forming a subdiagram of type A_5. Therefore, $|W| = 72 \cdot 6!$.

10.2.4 E_7



Let $H = \{x \in \mathbb{R}^8 \mid \sum x_i = 0\} \simeq \mathbb{R}^7$. Define

$$L = \mathrm{span}_{\mathbb{Z}}\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, \frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8)\} \cap H,$$

and let

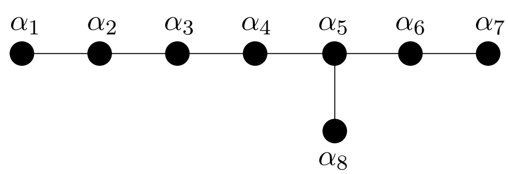
$$\Delta = \{v \in L \mid (v, v) = 2\} = \{e_i - e_j, \frac{1}{2}(e_i + e_j + e_k + e_l - e_p - e_q - e_r - e_s)\},$$

where all indices are distinct. Simple roots are $\Pi = \{\alpha_1 = e_1 - e_2, \ldots, \alpha_6 = e_6 - e_7,$ $\alpha_7 = \frac{1}{2}(-e_1 - e_2 - e_3 + e_4 + e_5 + e_6 + e_7 - e_8)\}$. The corresponding Weyl chamber is given by inequalities $C = \{x \in H \mid x_1 > x_2 > x_3 > x_4 > x_5 > x_6 > x_7, x_4 + x_5 + x_6 + x_7 > x_1 + x_2 + x_3 + x_8\}$, positive roots are $\{e_i - e_j\}$ for $i < j$, $\{\frac{1}{2}(e_i + e_j + e_k + e_l - e_p - e_q - e_r - e_8)\}$.

The highest root is $2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + 2\alpha_7 = e_1 - e_8$.

There are 126 roots, the highest root is orthogonal to simple roots forming a subdiagram of type D_6. Therefore, $|W| = 126 \cdot 6!2^5$.

10.2.5 E_8



Let $H = \{x \in \mathbb{R}^9 \mid \sum x_i = 0\} \simeq \mathbb{R}^8$. Define

$$L = \mathrm{span}_{\mathbb{Z}}\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, \frac{1}{3}(e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8 + e_9)\} \cap H,$$

and let

$$\Delta = \{v \in L \mid (v, v) = 2\} = \{e_i - e_j, \pm\frac{1}{3}(2(e_i + e_j + e_k) - (e_l + e_p + e_q + e_r + e_s + e_t))\},$$

where all indices are distinct. Simple roots are $\Pi = \{\alpha_1 = e_1 - e_2, \ldots, \alpha_7 = e_7 - e_8,$ $\alpha_8 = \frac{1}{3}(-e_1 - e_2 - e_3 - e_4 - e_5 + 2(e_6 + e_7 + e_8) - e_9)\}$. The corresponding Weyl chamber is given by inequalities $C = \{x \in H \mid x_1 > x_2 > x_3 > x_4 > x_5 > x_6 > x_7 > x_8, 2(x_6 + x_7 + x_8) > x_1 + x_2 + x_3 + x_4 + x_5 + x_9\}$, positive roots are $\{e_i - e_j\}$ for $i < j$, $\{\frac{1}{2}(2(e_i + e_j + e_k) - (e_l + e_p + e_q + e_r + e_s + e_9))\}$.

The highest root is $2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + 2\alpha_7 + 3\alpha_8 = e_1 - e_9$.

There are 240 roots, the highest root is orthogonal to simple roots forming a subdiagram of type E_7. Therefore, $|W| = 240 \cdot 126 \cdot 6!2^5$.

10.3 Bonus track: Finite Coxeter groups that are not Weyl groups

Remark. By dropping the condition $\langle \alpha \mid \beta \rangle \in \mathbb{Z}$ for $\alpha, \beta \in \Delta$ in the definition of a root system we get a definition of a *non-crystallographic* root system. In particular, the set of unit normal vectors to mirrors of all reflections of a finite reflection group satisfies the definition. Most of the results previously proved for root systems hold in these more general settings as well. In particular, everything in Section 9.3 (and HW 16.1) holds.

The construction of dihedral groups is clear. Here we show how to construct groups H_3 and H_4. The remaining part of Section 10.3 is NON-EXAMINABLE.

10.3.1 H_4

There are several explicit constructions of H_4. The one below follows Lusztig, Scherbak, and Moody–Patera.

Consider the Weyl group of type E_8 with simple reflections $s_1, \dots, s_4, t_1, \dots, t_4$, see Fig. 10.1. Denote $r_i = s_i t_i$, and consider group Γ generated by r_1, r_2, r_3, r_4. Clearly, elements r_i are involutions, and r_i commutes with r_j for $|i-j| > 1$. Also, since s_i commutes with t_j for $i, j \le 3$, $(r_1 r_2)^3 = (s_1 t_1 s_2 t_2)^3 = (s_1 s_2)^3 (t_1 t_2)^3 = e$, and similarly $(r_2 r_3)^3 = e$. Finally, $r_3 r_4 = s_3 t_3 t_4 s_4$ can be considered as an element of the group A_4, it is easy to check it has order 5 (see also Section 11).

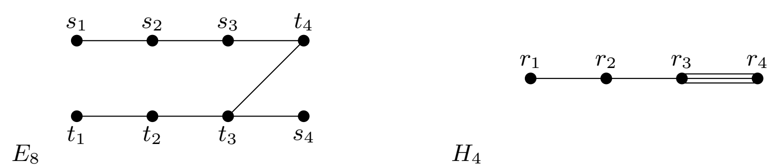


Figure 10.1: Construction of the group of type H_4 as a subgroup of the Weyl group E_8

Therefore, generators of Γ satisfy all relations of H_4, and thus there is a surjective homomorphism $H_4 \to \Gamma$.

To see that the kernel is trivial, we interpret r_i as a reflection in the ring of *quaternions* which is isomorphic to $\mathbb{R}^4$ as a vector space over $\mathbb{R}$.

Let $1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ be an orthonormal basis of $\mathbb{R}^4$. Define an operation of (non-commutative!) multiplication on the basis as follows:

$$1 \cdot x = x \cdot 1 = x, \quad , \boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1, \quad \boldsymbol{i} \cdot \boldsymbol{j} = -\boldsymbol{j} \cdot \boldsymbol{i} = \boldsymbol{k}, \quad \boldsymbol{j} \cdot \boldsymbol{k} = -\boldsymbol{k} \cdot \boldsymbol{j} = \boldsymbol{i}, \quad \boldsymbol{k} \cdot \boldsymbol{i} = -\boldsymbol{i} \cdot \boldsymbol{k} = \boldsymbol{j},$$

and then extend this to the whole $\mathbb{R}^4$ by linearity. We get the algebra of quaternions $\mathbb{H}$. An element of $\mathbb{H}$ has the form $x = (x_1, x_2, x_3, x_4) = x_1 + x_2\boldsymbol{i} + x_3\boldsymbol{j} + x_4\boldsymbol{k}$, $x_l \in \mathbb{R}$. There is a natural operation of conjugation: $\overline{x_1 + x_2\boldsymbol{i} + x_3\boldsymbol{j} + x_4\boldsymbol{k}} = x_1 - x_2\boldsymbol{i} - x_3\boldsymbol{j} - x_4\boldsymbol{k}$. Then the Euclidean norm $\|x\|^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$ can be written as $\|x\|^2 = x\bar{x}$. In particular, the unit sphere in $\mathbb{R}^4$ is $U = \{x \in \mathbb{H} \mid \|x\|^2 = 1\}$. Elements of U are called *unit quaternions*.

The standard dot product on $\mathbb{R}^4$ can also be written in these terms: $(x, y) = \frac{1}{2}(x\bar{y} + y\bar{x})$ (check this!).

Exercise. Let $x \in U$ be a unit quaternion, and $v \in \mathbb{H}$. Show that the reflection of v in x (as vectors in $\mathbb{R}^4$) is given by formula $r_x(v) = -x\bar{v}x$.

Denote $\varphi = 2\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{2}$, note that $\varphi^2 = \varphi + 1$. Consider all unit quaternions of types

$(\pm 1, 0, 0, 0)$ with all permutations;

$\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$;

$\frac{1}{2}(0, \pm 1, \pm\varphi, \pm(1-\varphi))$ with all *even* permutations.

Exercise. There are 120 quaternions of types above, and they form a group with respect to multiplication. We denote this set by I.

Now, let us construct a linear map from $\mathbb{R}^8$ to $\mathbb{H}$ as follows. Let α_i be simple roots of the root system E_8 corresponding to simple reflections s_i, and let α'_i be the simple roots corresponding to simple reflections t_i. Define unit quaternions $\beta_1, \dots, \beta_4$ as follows:

$$\beta_1 = \frac{1}{2}(\varphi - 1, -\varphi, 0, -1), \quad \beta_2 = \frac{1}{2}(0, \varphi - 1, -\varphi, 1), \quad \beta_3 = \frac{1}{2}(0, 1, \varphi - 1, -\varphi), \quad \beta_4 = \frac{1}{2}(0, -1, \varphi - 1, \varphi),$$

and then define a linear map $\mathbb{R}^8 \to \mathbb{H}$ by

$$f(\alpha_i) = \beta_i, \qquad f(\alpha'_i) = \varphi\beta_i$$

Exercise. The map f takes $s_i t_i$ to precisely r_{β_i}. In other words, for any $p \in \mathbb{R}^8$ we have

$$f(s_i t_i(p)) = r_{\beta_i}(f(p))$$

Exercise. The Gram matrix of vectors β_i is precisely the matrix defined by the Coxeter diagram H_4.

The two exercises above complete the construction. In particular, I is precisely the set of vectors orthogonal to mirrors of H_4, and f takes all 240 roots of E_8 to $I \cup \varphi I$.

More details can be found in the paper by R. V. Moody and J. Patera "Quasicrystals and icosians", J. Phys. A 26 (1993), 2829–2853.

To compute the order of the group observe that all simple reflections are conjugated to each other, and thus W acts transitively on all 120 elements of I. The vector $(1, 0, 0, 0)$ is orthogonal to $\beta_2, \beta_3, \beta_4$, and thus it belongs to $\overline{C}_0$ and its stabilizer is generated by r_2, r_3, r_4, which generate a group of type H_3 of order 120 (see next section). Therefore, $|W| = 120^2 = 14400$.

Remark. $W(H_4)$ is a symmetry group of two regular 4-dimensional polytopes, a 120-*cell* and a 600-*cell*.

10.3.2 H_3

The construction follows from the previous section: we consider subspace H of $\mathbb{R}^8$ spanned by roots α_i, α'_i for $i = 2, 3, 4$. Then we get $H \simeq \mathbb{R}^6$ with a root system of type D_6; H is taken by f to the subspace of $\mathbb{H}$ with zero first coordinate (such quaternions are called *pure*), the image of the root system D_6 is the set of vectors orthogonal to mirrors of H_3 (and their φ-multiples).

The order of the group can be computed as follows. There are 30 elements of I with zero first coordinate, which implies that there are 30 unit vectors orthogonal to mirrors (and 15 reflections), the group acts transitively on them. Vector $(0, \varphi, 1, \varphi - 1)$ is orthogonal to β_2 and β_4, so its stabilizer is generated by two commuting reflections r_2 and r_4. Therefore, $|W| = 30 \cdot 4 = 120$.

$W(H_3)$ is a symmetry group of two regular 3-dimensional polytopes, an icosahedron and a dodecahedron.

10.4 Affine Dynkin diagrams

Let Δ be an irreducible root system, $\Pi = \{\alpha_i\}$ is a set of simple roots, $\widetilde{\alpha}_0$ is the highest root. The root $\alpha_0 = -\widetilde{\alpha}_0$ is called the *lowest* root.

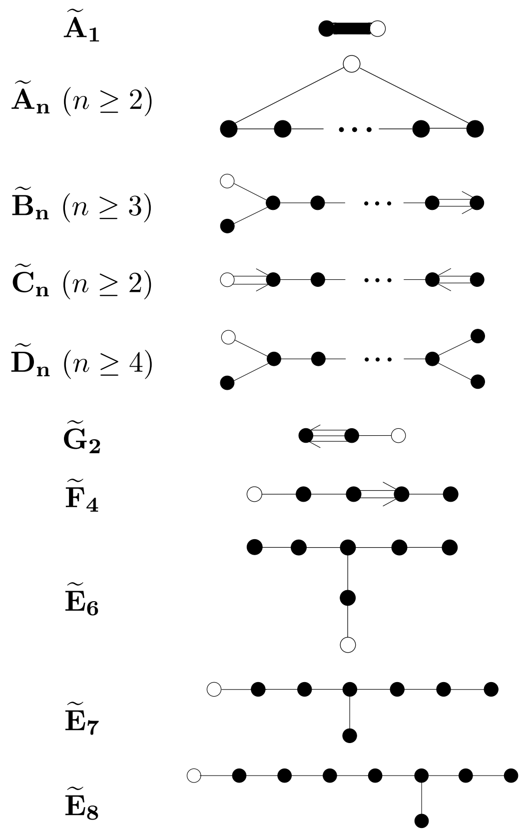
Definition 10.1. The set $\widehat{\Pi} = \{\alpha_0, \alpha_1, \dots, \alpha_n\}$ is an *extended* set of simple roots. The corresponding Dynkin diagram with $n+1$ nodes is an *extended Dynkin diagram*, or an *affine Dynkin diagram*.

The list of affine Dynkin diagrams is shown in Table 10.1.

Remark 10.2. We are now able to interpret the finiteness of the reflection game (see Section 6). The graph corresponds to a quadratic form, and the initial configuration is one basis vector α_i. A configuration is a linear combination $\sum c_i\alpha_i$. A vertex k is unstable if $(\alpha_k, \sum c_i\alpha_i) < 0$. A move in an unstable vertex k is a reflection of the configuration $\sum c_i\alpha_i$ in vector α_k. In case of Dynkin diagrams, the game ends when we get the highest root. If a (simply-laced) graph is not a Dynkin diagram, then it contains an affine Dynkin diagram as a subgraph; the game is infinite on any affine Dynkin diagram.

Definition 10.3. A *generalized Cartan matrix* $A = (a_{ij})$ is an integer $n \times n$ matrix satisfying the following properties: $a_{ii} = 2$, $a_{ij} \le 0$ for $i \ne j$, and $a_{ij} = 0$ implies $a_{ji} = 0$.

Table 10.1: **Affine Dynkin diagrams.** Connected affine Dynkin diagrams. Every diagram contains $n+1$ nodes. The lowest root is marked.



A is of *finite type* (respectively, *affine type* and *indefinite type*) if there exists a vector $u = (u_1, \dots, u_n) \in \mathbb{R}^n$ with all $u_i > 0$ such that Au also has all positive coordinates (respectively, $Au = 0$ and Au has all negative coordinates).

Remark 10.4. Indecomposable generalized Cartan matrices (GCM for short) can be of finite, affine or indefinite type only. If GCM is symmetric, then it is of finite type if and only if it is positive definite, and it is of affine type if and only if it is positive semidefinite with kernel of dimension one. Matrices of finite type correspond to connected Dynkin diagrams, and matrices of affine type correspond to connected affine Dynkin diagrams. See V. Kac, *Infinite-dimensional Lie algebras*, CUP, for more details.

Remark. Given an affine Dynkin diagram $\widetilde{\Sigma}$, one can define an *affine Weyl group* of $\widetilde{\Sigma}$ as a Coxeter group defined by $\widetilde{\Sigma}$. This group has a nice geometric interpretation: it is a group generated by *affine* reflections in the Euclidean space $\mathbb{E}^n$, i.e. a group generated by reflections in hyperplanes which may not pass through the origin.

Example. Consider the affine Dynkin diagram $\widetilde{A}_2$, it is a cycle of length 3. The corresponding affine Weyl group is $\widetilde{W} = \langle s_1, s_2, s_3 \mid s_i^2, (s_i s_j)^3 \rangle$. $\widetilde{W}$ acts on the plane as follows: s_1 and s_2 are usual reflections in lines $\alpha_1^\perp$ and $\alpha_2^\perp$ (where α_i are simple roots of A_2), and s_3 is a reflection with respect to the line $(v, \alpha_0) = -1$. The fundamental domain is a regular triangle.

In general, if an affine Dynkin diagram $\widetilde{\Sigma}$ is an extension of a Dynkin diagram Σ of a root system Δ, then the fundamental domain of $\widetilde{W}$ is a compact simplex bounded by n hyperplanes $\alpha_i^\perp$ (where α_i are simple roots of Δ), and a hyperplane $(v, \alpha_0) = -1$ (this simplex is called *alcove*). Group $\widetilde{W}$ is infinite and generated by reflections in the walls of the alcove.

Remark. If Δ is not simply-laced, one can see that the highest (and lowest) root is always long. However, the closure of the chamber C_0 also contains a short root. This leads to more extended Dynkin diagrams with the same affine Weyl group (the difference is in the lengths of roots only, i.e. in the directions of arrows in the extended Dynkin diagram).

11 Coxeter number

11.1 Coxeter element

Let (W, S) be an irreducible Coxeter system, where W is a finite Coxeter group of rank n, let $\{s_i = r_{\alpha_i}\}$ be simple reflections. We will assume by default that W is a Weyl group of a root system Δ with simple roots $\Pi = \{\alpha_i\}$, but everything in this section works for non-crystallographic groups as well.

Definition 11.1. An element $c = s_{i_1} \dots s_{i_n}$, where all indices are distinct, is called a *Coxeter element*.

In other words, a Coxeter element is a product of all simple reflections in some order. Clearly, there are $n!$ such elements, but some of them may coincide.

Example 11.2. If $\Delta = A_3$, then there are 6 Coxeter elements. However, as s_1 and s_3 commute, there are only four distinct ones: $s_1s_2s_3$, $s_3s_2s_1$, $s_2s_3s_1$ and $s_3s_1s_2$. Moreover, $s_1(s_1s_2s_3)s_1 = s_2s_3s_1 = s_2(s_3s_1s_2)s_2 = s_2(s_1s_3s_2)s_2 = s_2s_1(s_3s_2s_1)s_1s_2$, so all Coxeter elements in A_3 are conjugated to each other.

The same pattern can be seen in general.

Lemma 11.3. *All Coxeter elements are conjugated in W.*

Proof. First, observe that all cyclic shifts are conjugated: $s_{i_1}(s_{i_1}\dots s_{i_n})s_{i_1} = s_{i_2}\dots s_{i_n}s_{i_1}$ etc. Also, if two neighboring reflections commute, we may swap them without changing the element. We claim that using these two operations we can get *all* $n!$ Coxeter elements.

The proof is by induction on n. If $n = 1$ there is not much to prove, so assume the statement holds for $n-1$. Consider any Coxeter element c on letters $s_1, \dots, s_n$, we can assume s_n is a *leaf* of the Coxeter diagram of W, i.e. it commutes with all but one simple reflections. In particular, the Coxeter diagram of the subgroup $W_{S\setminus\{s_n\}}$ is connected, so we can apply the induction assumption.

By the induction assumption, any Coxeter element on letters $s_1, \dots, s_{n-1}$ can be taken to $s_1 \dots s_{n-1}$ by cyclic shifts and swapping commuting neighbors. Apply this procedure to c, we claim that the result is $s_1 \dots s_{n-1}$ with s_n inserted in one of n possible places. Indeed, s_n allows us to permute commuting s_i and s_j as s_n must commute with at least one of them, so we have either $s_is_ns_j = s_is_js_n = s_js_is_n$ or $s_is_ns_j = s_ns_is_j = s_ns_js_i$. Also, we can still perform cyclic shifts: if the word starts with $s_ns_i\dots$ and s_n does not commute with s_i, then we can do cyclic shift in s_n first and then in s_i.

Now, once we got $s_1\dots s_{n-1}$ with s_n inserted in one of n possible places, we can carry s_n either to the start of the word of to the end (the only s_i not commuting with s_n lies on one side of it), and then perform a cyclic shift if needed.

□

Corollary 11.4. *All Coxeter elements have the same order.*

Definition 11.5. The order of a Coxeter element is called the *Coxeter number* of Δ (or W) and is denoted by h.

Example 11.6. · $\Delta = A_2$, $(s_1s_2)^3 = e$, so $h = 3$.

- $\Delta = A_3$, $(s_1s_2s_3)^h = e$, how to find h? One way is to play with words: observe that h must be even (e.g. by the Deletion Condition), so try $h = 2$. We get

$$(s_1s_2s_3)^2 = s_1s_2s_3s_1s_2s_3 = s_1s_2s_1\,s_3s_2s_3 = s_2s_1\,s_2s_3s_2s_3 = s_2s_1s_3s_2 = s_2(s_1s_3)s_2 \neq e,$$

so $h \geq 4$. Now,

$$(s_1s_2s_3)^4 = (s_2(s_1s_3)s_2)^2 = s_2(s_1s_3)^2s_2 = e,$$

so $h = 4$.

Another way is to observe that for any A_{n-1} a Coxeter element $c = s_1\dots s_{n-1}$ can be written as a product of transpositions:

$$c = s_1\dots s_{n-1} = (12)(23)(34)\dots(n-1\,n) = (1\,2\,3\dots n-1\,n),$$

so it is a cycle of length n, and thus $h(A_{n-1}) = n$.

In general one can write matrices of the action of every simple reflection, take the product, and then compute eigenvalues.

Exercise. Compute $h(D_4)$.

11.2 Exponents

Observe that since all Coxeter elements are conjugated, they have the same characteristic polynomial, and thus the same eigenvalues. All eigenvalues are of the form ξ^m, where ξ is a primitive h-th root of unity, and $m \leq h-1$.

Definition 11.7. The *exponents* $m_1 \le m_2 \le \cdots \le m_n$ of W are all $m \in \mathbb{Z}_h$ such that ξ^m are eigenvalues of a Coxeter element. Exponents are counted with multiplicities.

Example 11.8. Let $\Delta = A_2$, then $c = s_1 s_2$ acts on $\mathbb{R}^2$ as a rotation by $2\pi/3$. Therefore, the eigenvalues of c are $\exp(\frac{2\pi i}{3})$ and $\exp(\frac{4\pi i}{3})$. Since $h = 3$, the primitive root of unity is $\xi = \exp(\frac{2\pi i}{3})$, so we get $m_1 = 1$ and $m_2 = 2$.

Exercise 11.9. Show that for $\Delta = A_n$ the exponents are $1, 2, \dots, n$.

Lemma 11.10. *$m_1 > 0$, i.e. 1 is not an eigenvalue of c.*

Proof. Let $c = s_1 \dots s_n$ and suppose there exists $v \in \mathbb{R}^n$ such that $c(v) = v$, $v \neq 0$. Then $s_1 \dots s_n v = v$ implies $s_2 \dots s_n v = s_1 v$. Let $s_i = r_{\alpha_i}$.

Note that $s_1 v = v + c_1\alpha_1$, and $s_2 \dots s_n v = v + \sum\limits_{i\ge 2} c_i \alpha_i$, so we obtain the equality $c_1\alpha_1 = \sum\limits_{i\ge 2} c_i\alpha_i$. However, simple roots are linearly independent, which implies that all $c_i = 0$. Therefore, $s_1 v = v$ and $s_2 \dots s_n v = v$. Applying the same argument $n-1$ times, we get $s_i v = v$ for every i, which implies that $(v, \alpha_i) = 0$ for every i. Since the dot product is not degenerate, this means that $v = 0$, so we came to a contradiction.

□

Lemma 11.11. *Let $m_1 \le \cdots \le m_n$ be exponents of W. Then $\sum m_i = \frac{nh}{2}$.*

Proof. The characteristic polynomial of c has real coefficients, so every eigenvalue has a complex conjugate one. Observe that conjugate to ξ^{m_i} is $\xi^{-m_i} = \xi^{h-m_i}$, so if m_i is an exponent of multiplicity l then $h - m_i$ is also an exponent of the same multiplicity. Due to Lemma 11.10, the only real eigenvalue can be $-1 = \xi^{h/2}$. Therefore, if we denote the multiplicity of $h/2$ by k, then the sum of exponents is $k\frac{h}{2} + \frac{n-k}{2}h = \frac{nh}{2}$. □

11.3 Coxeter plane

We now want to explore the geometry of the action of the Coxeter element and derive some interesting corollaries.

Since the Dynkin diagram Σ of Δ is a tree, we can color nodes in black and white such that neighboring nodes are of different colors (in general, graphs with this properties are called *bipartite*). Index white nodes $1, \dots, k$ and black nodes $k+1, \dots, n$. Consider $c' = s_1 \dots s_k$ and $c'' = s_{k+1} \dots s_n$.

Definition 11.12. Coxeter element $c = c'c''$ is called *bipartite.*

From now on we assume c is bipartite – we do not loose any generality as all Coxeter elements are conjugate.

Recall that a *dual* basis $\{\omega_i\}$ to Π is defined by $(\alpha_i, \omega_j) = 1$ if $i = j$ and $(\alpha_i, \omega_j) = 0$ otherwise.

Exercise 11.13. Let $A = (a_{ij}) = (\alpha_i, \alpha_j)$. Show that $\alpha_j = \sum\limits_i a_{ij}\omega_i$.

Corollary 11.14. *In the basis $\{\omega_i\}$, $A\omega_j = \alpha_j$.*

Lemma 11.15. *Matrix $A = (\alpha_i, \alpha_j)$ has a positive eigenvalue λ with a positive eigenvector $(\lambda_1, \dots, \lambda_n)$, i.e. $\lambda_i > 0$ for all i.*

Lemma follows from Perron-Frobenius Theorem (see HW 9.3).

Theorem 11.16. *Let $(\lambda_1, \ldots, \lambda_n)$ be a positive eigenvector of $A = (\alpha_i, \alpha_j)$. Denote*

$$\mu = \sum_{i=1}^{k} \lambda_i \omega_i \quad \text{and} \quad \nu = \sum_{j=k+1}^{n} \lambda_j \omega_j.$$

Then

(1) *The plane H spanned by μ and ν is preserved (not pointwise) by c' and c'', and thus by c.*

(2) *c' and c'' act on H as reflections.*

(3) *The order of c restricted to H is h.*

H is called the Coxeter plane *of c.*

Proof. The first goal is to compute $(\lambda - 1)\mu$ and $(\lambda - 1)\nu$.

Since $(\lambda_1, \ldots, \lambda_n)$ is an eigenvector of A, we have

$$A(\sum_{i=1}^{n} \lambda_i \omega_i) = \lambda(\sum_{i=1}^{n} \lambda_i \omega_i) = \sum_{i=1}^{n} \lambda \lambda_i \omega_i,$$

and by Exercise 11.13 we have

$$A(\sum_{i=1}^{n} \lambda_i \omega_i) = \sum_{i=1}^{n} \lambda_i A \omega_i = \sum_{i=1}^{n} \lambda_i \alpha_i$$

Take the scalar product of the vector above with α_i, $i \leq k$. We get

$$(\sum_{m=1}^{n} \lambda_m \alpha_m, \alpha_i) = (\sum_{m=1}^{n} \lambda \lambda_m \omega_m, \alpha_i) = \lambda \lambda_i.$$

On the other hand,

$$(\sum_{m=1}^{n} \lambda_m \alpha_m, \alpha_i) = \sum_{m=1}^{n} \lambda_m (\alpha_m, \alpha_i) = \sum_{m=1}^{n} \lambda_m a_{im} = \lambda_i + \sum_{j=k+1}^{n} \lambda_j a_{ij}$$

as $a_{im} = 0$ for $m \leq k$ unless $m = i$. Thus, we see that $\lambda \lambda_i = \lambda_i + \sum_{j=k+1}^{n} \lambda_j a_{ij}$, or, equivalently,

$$\lambda_i (\lambda - 1) = \sum_{j=k+1}^{n} \lambda_j a_{ij} \quad \text{for } i = 1, \ldots, k.$$

Therefore,

$$(\lambda - 1)\mu = (\lambda - 1) \sum_{i=1}^{k} \lambda_i \omega_i = \sum_{i=1}^{k} \lambda_i (\lambda - 1) \omega_i = \sum_{i=1}^{k} \left(\sum_{j=k+1}^{n} \lambda_j a_{ij} \right) \omega_i = \sum_{j=k+1}^{n} \lambda_j \left(\sum_{i=1}^{k} \alpha_{ij} \omega_i \right) =$$
$$= \sum_{j=k+1}^{n} \lambda_j \left(\sum_{m=1}^{n} a_{mj} \omega_i - \sum_{i=k+1}^{n} a_{ij} \omega_i \right) = \sum_{j=k+1}^{n} \lambda_j (\alpha_j - \omega_j) = \sum_{j=k+1}^{n} \lambda_j \alpha_j - \nu$$

since $a_{ij} = 0$ if both $i, j \geq k+1$ unless $i = j$. Similarly, we have $(\lambda - 1)\nu = \sum_{i=1}^{k} \lambda_i \alpha_i - \mu$. Summarizing, we get

$$\sum_{i=1}^{k} \lambda_i \alpha_i - (\lambda - 1)\nu = \mu \quad \text{and} \quad \sum_{j=k+1}^{n} \lambda_j \alpha_j - (\lambda - 1)\mu = \nu.$$

We will now use the obtained formulae to look how c' and c'' act on μ and ν. We have

$$c'(\nu) = r_{\alpha_1} \dots r_{\alpha_k} \left(\sum_{j=k+1}^{n} \lambda_j \omega_j \right) = \sum_{j=k+1}^{n} \lambda_j \omega_j = \nu$$

as $(\alpha_i, \omega_j) = 0$ for $i \leq k < j$. Similarly, $c''(\mu) = \mu$. Further,

$$c'(\mu) = c' \left(\sum_{i=1}^{k} \lambda_i \alpha_i - (\lambda - 1)\nu \right) = - \left(\sum_{i=1}^{k} \lambda_i \alpha_i \right) - (\lambda - 1)\nu = -(\mu + (\lambda - 1)\nu) - (\lambda - 1)\nu = -\mu - 2(\lambda - 1)\nu$$

since $c'(\alpha_i) = -\alpha_i$ for $i \leq k$. Similarly, $c'(\nu) = -\nu - 2(\lambda - 1)\mu$. Therefore, both c' and c'' take μ and ν to their linear combinations, so both c' and c'' preserve H.

Furthermore, restriction of c' on H is a non-trivial element of $O_2(\mathbb{R})$ with eigenvalue 1 (as $c'(\mu) = \mu$), so the other eigenvalue must be -1, and thus c' acts on H as a reflection. Similarly, the same is true for c''. In particular, the restriction of c on H is a rotation.

Therefore, we have proved (1) and (2). To prove (3), observe first that every ω_i is a 1-dimensional face of the initial chamber C_0 (cf. the proof of Lemma 9.13), and thus $\mu + \nu = \sum_{i=1}^{n} \lambda_i \omega_i \in C_0$ as all $\lambda_i > 0$. In particular, we see that $H \cap C_0 \neq \emptyset$. Take any $p \in H \cap C_0$. If $c^m(p) = p$, then $c^m = e$ by Theorem 7.7, which implies that the order of c restricted to H is precisely h, so the proof is complete.

□

Lemma 11.17. *Let $m_1 \leq \cdots \leq m_n$ be the exponents, let h be the Coxeter number. Then*

(1) $m_1 = 1$, $m_n = h - 1$;

(2) $|\Delta^+| = \frac{nh}{2}$.

Proof. Observe that any positive linear combination $a\mu + b\nu$ belongs to C_0, i.e., the angle in H formed by rays $\mathbb{R}_+\mu$ and $\mathbb{R}_+\nu$ lies in C_0. This implies that this angle is not intersected by any mirror of W. Also, the reflections (on H) c' and c'' in $\mathbb{R}\nu$ and $\mathbb{R}\mu$ respectively generate a dihedral group of order $2h$. We want to show that the angle formed by rays $\mathbb{R}_+\mu$ and $\mathbb{R}_+\nu$ is equal π/h – this will imply that $m_1 = 1$.

Suppose that $\angle(\mathbb{R}_+\mu, \mathbb{R}_+\nu) = \frac{m\pi}{h}$, where $m \geq 2$. Then the intersection $H \cap C_0$ is an angle of size at least $2\pi/h$. The orbit of $H \cap C_0$ under the group generated by c' and c'' consists of $2h$ copies of it, and no two copies can overlap (otherwise they would need to coincide as each angle belongs to one chamber only, which, in its turn, would mean that the corresponding elements of the dihedral group must be equal, which cannot happen). This immediately leads to a contradiction.

Therefore, $m_1 = 1$, and thus the complex conjugate eigenvalue gives $m_n = h - 1$, so (1) is proved.

To prove (2), observe first that mirrors of W can intersect H along copies of $\mathbb{R}\mu$ and $\mathbb{R}\nu$ under the dihedral group. There are $2h$ angles formed by these lines, so there are h lines in total. Our plan is to count the number of mirrors intersecting H along $\mathbb{R}\mu$ and $\mathbb{R}\nu$.

Denote $H_i = \alpha_i^\perp$. By definition, $\mathbb{R}\mu = (H_{k+1} \cap \cdots \cap H_n) \cap H$. We want to prove the following claim: no other mirror of W contains $\mathbb{R}\mu$.

Suppose there is some $H_\alpha \cap H = \mathbb{R}\mu$, $\alpha = \sum_i b_i\alpha_i$, $b_i \geq 0$. This implies $(\alpha, \mu) = 0$, so we have

$$0 = (\alpha, \mu) = \left(\sum_{i=1}^{n} b_i\alpha_i, \sum_{i=1}^{k} \lambda_i\omega_i\right) = \sum_{i=1}^{k} b_i\lambda_i.$$

As all $\lambda_i > 0, b_i \geq 0$, this implies that $b_i = 0$ for $i = 1, \dots, k$, so $\alpha = \sum_{i=k+1}^{n} b_i\alpha_i$. However, all α_i for $i = k+1, \dots, n$ are mutually orthogonal, and thus by HW 18.2, $\alpha = \alpha_i$ for some $i > k$, so the claim is proved.

Therefore, $\mathbb{R}\mu$ is contained in $n-k$ mirrors of W. Similarly, $\mathbb{R}\nu$ is contained in k mirrors of W. The copies of $\mathbb{R}\mu$ and $\mathbb{R}\nu$ under the action of the dihedral group generated by c' and c'' are also contained in $n-k$ and k mirrors respectively. If h is even, then there are $h/2$ lines of each of the two types, so $|\Delta^+| = \frac{h}{2}(n-k) + \frac{h}{2}k = \frac{nh}{2}$. If h is odd, then $\mathbb{R}\mu$ and $\mathbb{R}\nu$ are equivalent under the action of the dihedral group, and thus $k = n-k$, i.e. $k = n/2$. Therefore, $|\Delta^+| = hn/2$ as well.

□

Combining Lemmas 11.11 and 11.17, we get the following corollary.

Corollary 11.18. *The number of positive roots in a root system is equal to the sum of exponents.*

Exercise. If h is even, then the longest element g_0 of W is precisely $c^{h/2}$.

11.4 More about Coxeter number and exponents

We list here more facts without proofs.

Theorem 11.19. *Let $m_1 \leq \cdots \leq m_n$ be the exponents, define partition $\mu = (m_n, \dots, m_1) \vdash |\Delta^+|$. Let $\lambda = (l_1, \dots, l_{m_n})$ be the dual Young diagram. For $\alpha = \sum c_j\alpha_j \in \Delta^+$ let the height $\operatorname{ht}\alpha$ denote $\sum c_j$. Then $l_i = \#\{\alpha \in \Delta^+ \mid \operatorname{ht}\alpha = i\}$.*

Example 11.20. Consider $\Delta = A_n$. We know the exponents are $m_i = i$. This implies $l_i = n+1-i$. Indeed, roots of height i are precisely $e_j - e_{j+i} = \alpha_j + \cdots + \alpha_{j+i-1}$, so $j = 1, \dots, n+1-i$, and thus the number of roots of height i is precisely $n+1-i$.

Theorem 11.21. *Let h be the Coxeter number of Δ, and let positive integer $m < h$ be coprime with h. Then m is an exponent of Δ.*

Idea of the proof. Consider a *cyclotomic polynomial*

$$\Phi_h(z) = \prod_{\substack{\gcd(k,h)=1 \\ 1 \leq k \leq h-1}} (z - e^{\frac{2\pi i}{h}k}),$$

it is a well known fact that $\Phi_h(z) \in \mathbb{Z}[z]$ and $\Phi_h(z)$ is irreducible. Note that $\Phi_h(z)$ has a common root $e^{2\pi i/h}$ with the characteristic polynomial of c (since 1 is an exponent, see Lemma 11.17). Therefore, $\Phi_h(z)$ divides the characteristic polynomial of c, which implies the theorem.

□

Example 11.22. Assuming we know that the Coxeter number of B_5 is equal to 10, let us find the exponents. By Theorem 11.21, $1, 3, 7, 9$ are exponents. The sum of all five exponents is $nh/2 = 25$ (Lemma 11.11), so the fifth exponent is $25 - (1+3+7+9) = 5$, thus exponents are $1, 3, 5, 7, 9$.

Alternatively, to find the remaining exponent one could notice that since it does not come in pair, it should be equal to $h/2 = 5$.

Theorem 11.23. *The order of the Weyl group can be computed as follows:*

$$|W| = (1+m_1)(1+m_2)\dots(1+m_n)$$

The proof (following R. Steinberg, *Finite reflection groups*, Trans. Amer. Math. Soc. 91 (1959)) of the following theorem is contained in HW 19.3.

Theorem 11.24. *Let $\widetilde{\alpha}_0$ be the highest root. Then* $\operatorname{ht}\widetilde{\alpha}_0 + 1 = h$.

The next section is NON-EXAMINABLE.

12 Coxeter-Catalan combinatorics

We will now define various objects on Weyl groups which will generalize the ones we have seen in the first term. When restricted to the symmetric group, these will coincide with what we have already seen.

12.1 Inversions

We will assume by default that W is a Weyl group of a root system Δ with simple roots $\Pi = \{\alpha_i\}$, but everything in this section works for non-crystallographic groups as well.

Definition 12.1. Let W be a Weyl group of a root system Δ, $\Pi = \{\alpha_i\}$ are simple roots. Given $w \in W$, $\alpha \in \Delta^+$ is an *inversion* of w if $w(\alpha) \in \Delta^-$. Denote by $\operatorname{Inv}(w)$ the set of inversions of w.

Example. Let $\Delta = A_2$, $w = s_1 s_2$. There are three positive roots, and $w(\alpha_1) = \alpha_2$, $w(\alpha_2) = -\alpha_1 - \alpha_2$, and $w(\alpha_1+\alpha_2) = -\alpha_1$. Therefore, $\operatorname{Inv}(w) = \{\alpha_2, \alpha_1+\alpha_2\}$. In particular, we see that $|\operatorname{Inv}(w)| = l(w)$.

Lemma 12.2. *Let W be a Weyl group. Then $|\operatorname{Inv}(w)| = l(w)$ for any $w \in W$.*

Proof. We know that the length of w is equal to the number of reflections whose mirrors separate the initial Weyl chamber C_0 from wC_0 (see Remark 8.11). Take any $v \in C_0$ and $\alpha \in \Delta^+$, then the mirror of r_α separates C_0 from wC_0 if and only if $(\alpha, w(v)) < 0$, which is equivalent to $(w^{-1}(\alpha), v) < 0$, i.e. $\alpha \in \operatorname{Inv}(w^{-1})$. Therefore, $R(w) = \operatorname{Inv}(w^{-1})$, so $l(w) = |\operatorname{Inv}(w^{-1})|$. Since $l(w) = l(w^{-1})$, we get $|\operatorname{Inv}(w^{-1})| = l(w^{-1})$ for any $w \in W$, which implies the statement of the lemma.

□

Remark. Another proof of the lemma can be found in HW 16.1.

As a corollary of the proof, we obtain the following statement.

Corollary 12.3. $\operatorname{Inv}(w) = \{\alpha \in \Delta^+ \mid\ r_a \in R(w^{-1})\}$.

Recall that for a symmetric group we had another definition of inversion (see Def. 3.2): given a permutation $w = w_1 \dots w_n$, an inversion is a pair $i < j$ such that $w_i > w_j$.

Example 12.4. Let $W = S_4$, i.e. $\Delta = A_3$. Take $w = s_1 s_2 s_3 = 2\,3\,4\,1$. Then $w^{-1} = 4\,1\,2\,3$, and the inversions of w^{-1} are $(1,2)$, $(1,3)$ and $(1,4)$. Now, the R-sequence of w is $\{s_1 = (12), s_1s_2s_1 = (12)(23)(12) = (13), s_1s_2s_3s_2s_1 = (12)(23)(34)(23)(12) = (12)(24)(12) = (14)\}$, so we see that the inversion set of w^{-1} coincides with the R-sequence of w. In other words, for w^{-1} the two definitions of inversion coincide.

Theorem 12.5. *The two definitions of inversions for symmetric group coincide.*

Proof. Let $w = w_1 w_2 \dots w_n \in S_n$. An inversion of w^{-1} is a pair (w_j, w_i) such that $i < j$ and $w_i > w_j$. The symmetric group $W = S_n$ acts on $\mathbb{R}^n$ by permutations of coordinates, i.e. w maps $e_i \mapsto e_{w_i}$. If $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, then $w(x)_{w_i} = x_i$. The Weyl chamber C_0 is given by equations $x_1 > x_2 > \dots > x_n$.

Let $x \in C_0$. If $i < j$ and $w_i > w_j$, then $x_i > x_j$ and $x_{w_i} < x_{w_j}$, which is equivalent to $w(x)_{w_i} = x_i > x_j = w(x)_{w_j}$, i.e. $w(x)_{w_i} > w(x)_{w_j}$, while $w_i > w_j$. This is equivalent to the hyperplane $y_{w_i} = y_{w_j}$ separating C_0 from wC_0, i.e. to $(w_j, w_i) \in R(w)$, and thus to $(w_j, w_i) \in \mathrm{Inv}\,(w^{-1})$ due to Cor. 12.3. □

Remark. Theorem 12.5 shows that Inv can be considered as a generalization of the inversion statistics to all Weyl groups (and, in fact, to all finite Coxeter groups).

Corollary 12.3 can be reformulated as follows.

Corollary 12.6. *Let $w = s_1 \dots s_k$ be a reduced expression, $w = r_k \dots r_1$, where $\{r_1 = r_{\beta_1}, \dots, r_k = r_{\beta_k}\}$ is the R-sequence of w. Then $\{\beta_i\} = \mathrm{Inv}\,(w^{-1})$.*

12.2 W-Catalan numbers

Given a Coxeter system (W, S), define a number

$$N(W) = \prod_{i=1}^{n} \frac{m_i + h + 1}{m_i + 1},$$

where h is the Coxeter number, m_i are exponents, and n is the rank of W. We also write $N(W) = N(\Delta)$ if W is the Weyl group of root system Δ.

Example 12.7. Let $W = S_{n+1}$. Then $h = n + 1$, $m_i = 1, \dots, n$, so

$$N(S_{n+1}) = \frac{(1 + (n+2))(2 + (n+2)) \dots (n + (n+2))}{(1+1)(2+1) \dots (n+1)} = \frac{(2n+2)!}{(n+2)!(n+1)!} = \frac{1}{n+2}\binom{n+2}{n+1} = C_{n+1}$$

Remark. $N(W)$ can be understood as a generalization of the Catalan numbers to other Weyl groups. In other words, objects counted by Catalan numbers are of "A_n type", and they can often be generalized to other root systems/Weyl groups. We consider some examples below.

12.3 Non-nesting partitions

Lemma 12.8. *There is a bijection between non-nesting partitions of $[n+1]$ and antichains in the root poset of A_n.*

Proof. The map from the set of antichains to the set of partitions is constructed as follows: given an antichain, draw an arc (i, j) for every root $e_i - e_j$ it contains, and define a partition by this arc diagram. The resulting partition is indeed non-nesting: if $i < j < k < l$, then $e_i - e_l > e_j - e_k$, so nesting arcs cannot appear. The map is injective by construction. Also, any non-nesting partition can be obtained: take any admissible arc diagram, then any two arcs define incomparable roots.

□

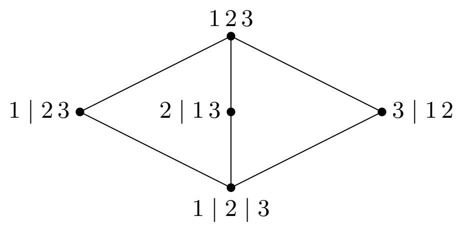
Therefore, we can consider antichains in the root poset of a root system Δ as a generalization of non-nesting partitions.

Theorem 12.9 (Athanasiadis, Postnikov, Reiner)**.** *The number of antichains in the root poset of a root system Δ is $N(\Delta)$.*

12.4 Non-crossing partitions NC_n

Define a partial order on the set NC_n of non-crossing partitions of $[n]$: we say $P_1 \leq P_2$ if every block of P_1 is contained in some block of P_2.

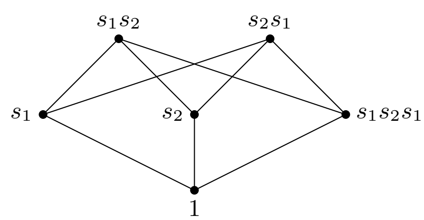
Example. Let $n = 3$. Then the Hasse diagram of the order is



The poset NC_n with the order as above is called an NC_n*-lattice* (check it is indeed a lattice!).

Now consider a Weyl group W, and define an *absolute order* on W as follows. Let R be the set of reflections, and for $w \in W$ let $L(w)$ denote the "R-length" of w, i.e. the minimal k such that $w = r_1 \dots r_k$, $r_i \in R$. We say $w_1 \preceq w_2$ if $w_2 = w_1 w$ and $L(w_2) = L(w_1) + L(w)$.

Example. Let $\Delta = A_n$, $n = 2$. The the Hasse diagram of the absolute order looks as follows.



Definition 12.10. Let P be a poset with partial order $\leq$. Given $p_1, p_2 \in P$, an *interval* $[p_1, p_2]$ is defined by

$$[p_1, p_2] = \{p \in P \mid p_1 \leq p \leq p_2\}$$

Lemma 12.11. *Let $W = S_n$, c is a Coxeter element. Then the interval $[1, c]$ in the absolute order is isomorphic to the NC_n-lattice.*

Theorem 12.12 (Reiner, Bessis)**.** *The number of elements in $[1, c]$ is equal to $N(\Delta)$.*

Therefore, the interval $[1, c]$ in the absolute order of any Weyl group W can be considered as a generalization of the NC_n-lattice.

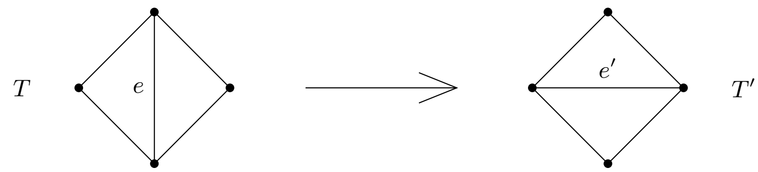
There is a more geometric way to interpret non-crossing partitions for Weyl groups which is due to Reading. Let $w \in [1, c]$, $w = r_1 \dots r_k$ reduced in the R-alphabet. Take the group Γ generated by $r_1, \dots, r_k$ – this is a parabolic subgroup (cf HW 17.2; note: this statement requires a proof). Then the absolute order is equivalent to containment of the corresponding parabolic subgroups. Take the smallest orbit O of W (e.g., for $\Delta = A_n$ the largest stabilizer is A_{n-1}, so the smallest orbit O has $n+1$ elements). Project the orbit to the Coxeter plane (defined by c) orthogonally (e.g., for A_n we get a regular $(n+1)$-gon). Then O is split into orbits of Γ, every orbit is precisely a block of the partition corresponding to w.

Details of the construction can be found in N. Reading, *Noncrossing partitions, clusters and the Coxeter plane*, Sém. Lothar. Combin. 63 (2010), Art. B63b.

12.5 Associahedra

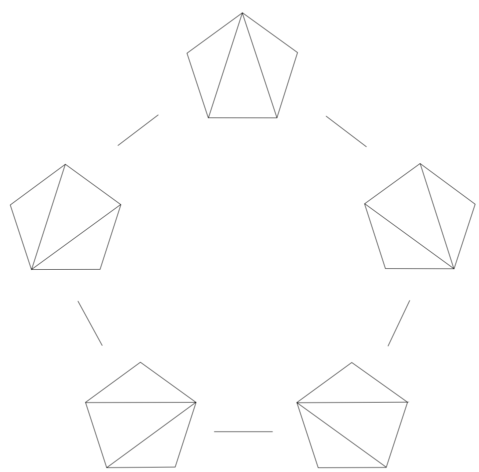
Recall: the number of triangulations of an $(n+3)$-gon is C_{n+1}.

Definition 12.13. Given a triangulation T and an edge e of T, a *flip* of T in e produces a new triangulation T' as follows: it substitutes e with the other diagonal of the corresponding quadrilateral, see the picture below.



An *exchange graph* of triangulations of an $(n+3)$-gon has all triangulations as its vertices, and two vertices are connected by an edge if the corresponding triangulations are related by a flip.

Example. Let $n = 2$, then there are five triangulations of a pentagon.



For $n = 3$, there are 14 triangulations of a hexagon, see Fig. 12.1.

The exchange graph is a 1-skeleton of an n-dimensional polytope which is called an *associahedron* (or Stasheff polytope).

One relation of the associahedron to the A_n root system can be seen from Theorem 12.15.

Definition 12.14. Let $W = S_{n+1}$, let C be a Weyl chamber of root system A_n, and let $x \in C$. The convex hull of the orbit Wx is a convex n-polytope called *permutohedron*.

Theorem 12.15 (Tonks)**.** *An n-dimensional associahedron can be obtained from an n-dimensional permutohedron by contraction of some edges.*

We will now make the relation of the associahedron to root system A_n more precise.

Definition 12.16. Let Δ be a root system, $\Pi = \{\alpha_i\}$ are simple roots. The set of *almost positive roots* $\Delta_{\geq -1}$ is defined by

$$\Delta_{\geq -1} = \Delta^+ \sqcup \{-\alpha_i \mid \alpha_i \in \Pi\}$$

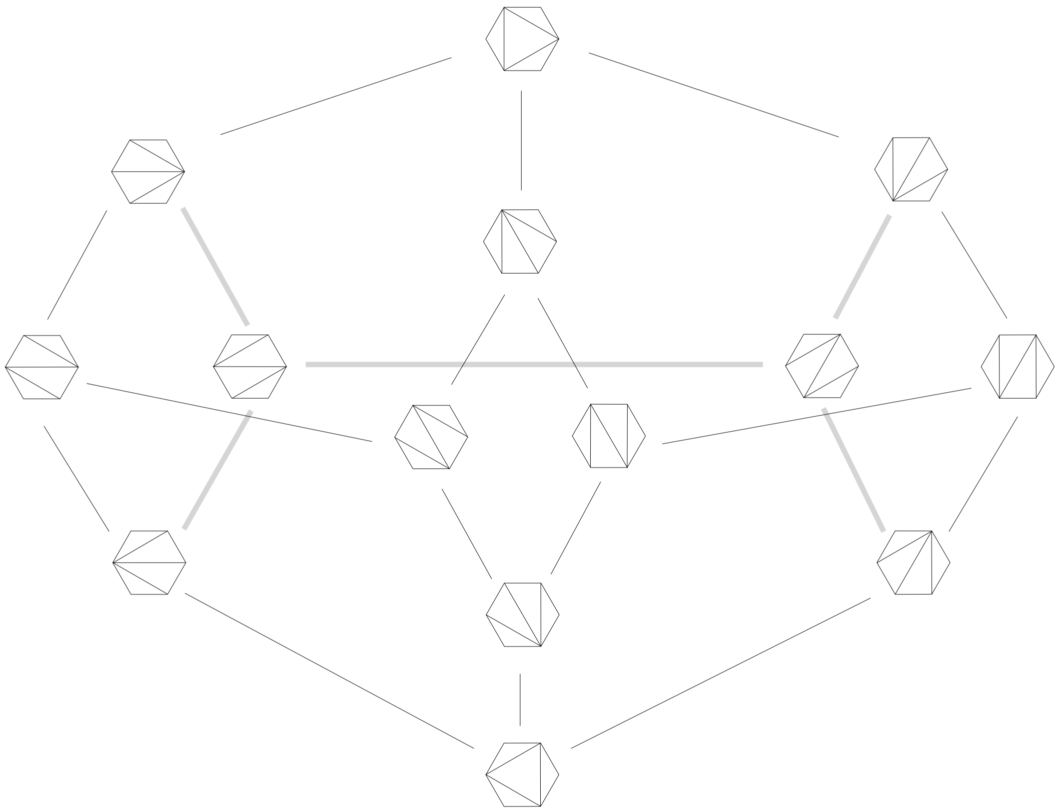


Figure 12.1: Exchange graph of triangulations of a regular hexagon.

Definition 12.17. Let $c = c'c''$ be a bipartite Coxeter element of a root system Δ, where $c' = \prod_{i=1}^{k} s_i$ and $c'' = \prod_{i=k+1}^{n} s_i$. Define $\tau', \tau'' : \Delta_{\geq -1} \to \Delta_{\geq -1}$ by

$$\tau'(\alpha) = \begin{cases} \alpha, & \alpha = -\alpha_i,\ i \geq k+1 \\ c'(\alpha) & \text{otherwise} \end{cases} \qquad \text{and} \qquad \tau''(\alpha) = \begin{cases} \alpha, & \alpha = -\alpha_i,\ i \leq k \\ c''(\alpha) & \text{otherwise} \end{cases}$$

Exercise. Check that τ', τ'' are involutions.

Example 12.18. Let $\Delta = A_2$, $c' = s_1 = r_{\alpha_1}$, $c'' = s_2 = r_{\alpha_2}$. Then τ' and τ'' act on $\Delta_{\geq -1}$ as follows.

$$\overset{\overset{\tau''}{\curvearrowleft}}{-\alpha_1} \overset{\tau'}{\longleftrightarrow} \alpha_1 \overset{\tau''}{\longleftrightarrow} \alpha_1 + \alpha_2 \overset{\tau'}{\longleftrightarrow} \alpha_2 \overset{\tau''}{\longleftrightarrow} \overset{\overset{\tau'}{\curvearrowright}}{-\alpha_2}$$

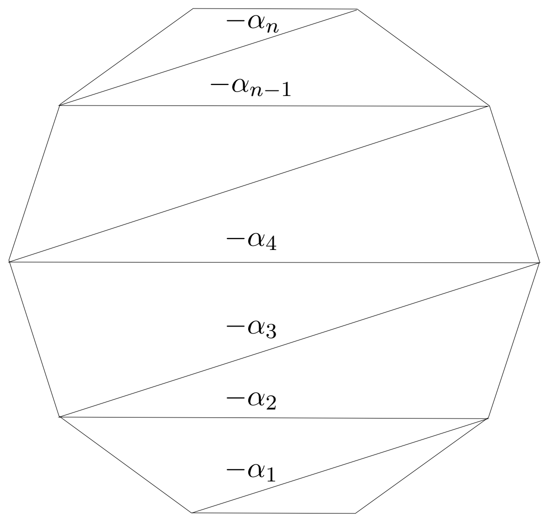
Theorem 12.19 (Fomin-Zelevinsky)**.** *There is a unique binary relation (called* compatibility*) on $\Delta_{\geq -1}$ satisfying the following two properties:*

(a) It is invariant with respect to the action of τ' and τ'', i.e. α is compatible with β if and only if $\tau'(\alpha)$ is compatible with $\tau'(\beta)$, and if and only if $\tau''(\alpha)$ is compatible with $\tau''(\beta)$;

(b) $-\alpha_i$ is compatible with β if and only if the expression of β as a linear combination of α_j does not contain α_i.

Example 12.20. In A_2, there are five pairs of compatible elements: $(-\alpha_1, -\alpha_2), (-\alpha_1, \alpha_2), (-\alpha_2, \alpha_1), (\alpha_1, \alpha_1 + \alpha_2), (\alpha_2, \alpha_1 + \alpha_2)$.

Example 12.21. Let $\Delta = A_n$. Observe that the number of elements in $\Delta_{\geq -1}$ is $\frac{n(n+1)}{2} + n = \frac{n(n+3)}{2}$, which is equal to the number of diagonals of $(n+3)$-gon. Consider a "staircase" triangulation of an $(n+3)$-gon (see below), and assign to its edges negative simple roots as follows.



For every other diagonal d not in the triangulation, assign to it a root $\alpha_i + \cdots + \alpha_j$ if d intersect precisely edges of the triangulation with assigned roots $-\alpha_i, \ldots, -\alpha_j$. Then the roots are compatible if and only if the corresponding diagonals do not intersect. In particular, all triangulations are precisely maximal sets of mutually compatible roots.

This implies that the associahedron can be considered as follows: the vertices are maximal compatible sets of roots, and faces of dimension $n-k$ are collections of k compatible roots. For example, the 2-dimensional faces of 3-dimensional associahedron shown in Figure 12.1 correspond to roots from $\Delta_{\geq -1}$, or, equivalently, to diagonals: given a face, there is a unique diagonal which is present in every triangulation corresponding to a vertex of that face; in Figure 12.1, quadrilateral faces correspond to long diagonals of the hexagon, and pentagonal faces correspond to short diagonals.

Theorem 12.22 (Fomin-Zelevinsky)**.** *For every root system Δ of rank n, all maximal compatible sets in $\Delta_{\geq -1}$ have cardinality n. For every maximal compatible set Σ and any $\alpha \in \Sigma$ there exists a unique $\beta \notin \Sigma$ such that $(\Sigma \setminus \alpha) \cup \{\beta\}$ is maximal compatible. The number of maximal compatible sets is equal to $N(\Delta)$.*

Theorem 12.22 allows us to define an exchange graph of maximal compatible sets for any Weyl group. These are 1-skeletons of convex polytopes called *generalized associahedra*. Faces of dimension $n-k$ are precisely collections of k compatible roots. For details, see the lecture notes *Root systems and generalized associahedra* by Fomin and Reading, and references therein.