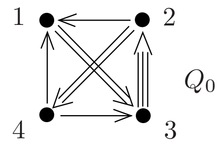
Exercises - 2

1. Let Q_0 be the following quiver:

   Find $Q_1 = \mu_1(Q_0)$, the mutation μ_1 of Q_0 in the 1st vertex.



2. Find $Q_2 = \mu_2(Q_1)$.
   Hint: you don't need any additional computations!

3. Given initial variables x_1, x_2, x_3, x_4 attached to the vertices of Q_0, find the variables attached to the vertices of Q_2.

- **Definition.** A *Somos sequence* of order k (or a *Somos-k sequence*) is a recurrent sequence defined by

$$a_n = \frac{\sum_{j=1}^{[k/2]} a_{n-j}a_{n-(k-j)}}{a_{n-k}}$$

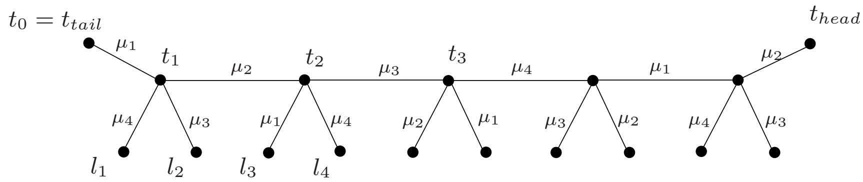
  with $a_j = 1$ for $j = 1, \dots, k$.
  In particular, the *Somos-4 sequence* is defined by $a_n a_{n-4} = a_{n-1}a_{n-3} + a_{n-2}^2$.

4. Show that applying the sequence of mutations

$$\mu_1, \mu_2, \mu_3, \mu_4, \mu_1, \mu_2, \mu_3, \dots$$

   to the initial seed $t_0 = (Q_0, \mathbf{x})$, with $\mathbf{x} = (x_1, x_2, x_3, x_4) = (1, 1, 1, 1)$
   one obtains all elements of the Somos-4 sequence.

- Consider the **caterpillar**:



- **Notation**. Denote by $\mathcal{L}(s)$ the ring of Laurent polynomials in the variables of a seed s.
  Denote also $x_i(s)$ the i-th variable in a seed s.

5. Show that if $x_i(t_j) \in \mathcal{L}(t_0)$ for all $i = 1, 2, 3, 4$, $j \in \mathbb{Z}_+$ then all elements of the Somos-4 sequence are integers.

6. Check that $x_i(t_1) \in \mathcal{L}(t_0)$ and $x_i(l_3) \in \mathcal{L}(t_0)$ for all $i = 1, \dots, 4$,
   (i.e. all variables in seeds t_1 and l_3 are Laurent polynomials in variables of the seed t_0).

7. Show that $gcd(x_1(t_1), x_1(l_3)) = gcd(x_1(t_1), x_2(t_2)) = 1$ (as elements of $\mathcal{L}(t_0)$).

8. In this question we want to prove that all variables attached to the spine vertices t_j of the caterpillar are Laurent polynomials in variables of t_0. We will use inductive arguments. Let x be a variable of the seed t_{head}. Suppose inductively that we know $x \in \mathcal{L}(s)$ for all seeds s in the caterpillar lying closer to t_{head} than t_0. We will prove that $x \in \mathcal{L}(t_{head})$ as well as that $x \in \mathcal{L}(l_1)$ and $x \in \mathcal{L}(l_2)$.

   (a) Use the inductive assumption and Question 6 to see that $x = \frac{f}{x_1(t_1)^a}$ for some $f \in \mathcal{L}(t_0)$ and $a \in \mathbb{Z}_{\geq 0}$.
   Show similarly that $x = \frac{g}{x_2(t_2)^b x_1(l_3)^c}$ for some $g \in \mathcal{L}(t_0)$ and $b, c \in \mathbb{Z}_{\geq 0}$.

   (b) Use Question 7 to see that $x \in \mathcal{L}(t_0)$.

   (c) Similarly to (a) and (b), use the induction assumption as well as the analogues of Questions 6 and 7 to prove that $x \in \mathcal{L}(l_1)$ and $x \in \mathcal{L}(l_2)$.
   Hint: you need to consider sequences of mutations $s_0 \overset{\mu_i}{-} s_1 \overset{\mu_j}{-} s_2 \overset{\mu_i}{-} s_3$ and show that $x_i(s_3) \in \mathcal{L}(s_0)$ and that $gcd(x_i(s_3), x_i(s_1)) = gcd(x_i(s_3), x_j(s_2)) = 1$.

   (d) Prove the base of the induction.