Jacobs University
School of Engineering and Science

Fall Term 2009

ESM 1B, Homework 8

Due Date: 14:00 Thursday, November 12, 2009.

Explain your answers! Problems marked ($\star$) are bonus ones.

8.1. Consider the curve in the plane given by the equation $y = f(x)$. Find the curvature of this curve at point $(x, f(x))$ in terms of the derivatives of f.

8.2. Let f, g be scalar fields, and let $\vec{u}$, $\vec{v}$ be vector fields. Prove the following formulas.

(a) $\operatorname{curl}(\operatorname{grad} f) = 0$;

(b) $\operatorname{curl}(f\vec{u}) = (\operatorname{grad} f) \times \vec{u} + f \operatorname{curl} \vec{u}$;

(c) $\operatorname{div}(\vec{u} \times \vec{v}) = \vec{v} \cdot (\operatorname{curl} \vec{u}) - \vec{u} \cdot (\operatorname{curl} \vec{v})$;

(d) $\operatorname{div}(\operatorname{grad} f \times \operatorname{grad} g) = 0$.

8.3. Let $(\rho, \vartheta, \varphi)$ be spherical coordinates:

$$x = \rho \sin\vartheta \cos\varphi, \quad y = \rho \sin\vartheta \sin\varphi, \quad z = \rho \cos\vartheta.$$

Let $\vec{e}_\rho$, $\vec{e}_\vartheta$ and $\vec{e}_\varphi$ be the unit vectors of the same direction as the partial derivatives

$$\frac{\partial \vec{r}}{\partial \rho}, \quad \frac{\partial \vec{r}}{\partial \vartheta}, \quad \frac{\partial \vec{r}}{\partial \varphi}.$$

Prove that the gradient of a scalar field f has the following expression in the cylindrical coordinates:

$$\nabla f = \frac{\partial f}{\partial \rho}\vec{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \vartheta}\vec{e}_\vartheta + \frac{1}{\rho \sin\vartheta}\frac{\partial f}{\partial \varphi}\vec{e}_\varphi.$$

You may use the textbook but please give a detailed and complete computation.

8.4. Deduce from Maxwell equations that

$$\operatorname{div} \mathbf{j} = -\frac{\partial \rho}{\partial t}.$$

8.5. ($\star$) Let $f = 1/r$, where $r = \sqrt{x^2 + y^2 + z^2}$. Write down the expression for vector field $\vec{u} = \operatorname{grad} f$

(a) in Cartesian coordinates;

(b) in cylindrical coordinates;

(c) in spherical coordinates.

(d) Compute $\operatorname{div} \vec{u}$ and $\operatorname{curl} \vec{u}$.