Jacobs University
School of Engineering and Science

Fall Term 2009

ESM 1B, Homework 9

Due Date: 14:00 Thursday, November 26, 2009.

Explain your answers! Problems marked ($\star$) are bonus ones.

9.1. Evaluate the following line integrals over the circle $C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$:

(a) $\displaystyle\int_C (x + y)\, ds$; (b) $\displaystyle\int_C \vec{u} \cdot d\vec{r}$, where $\vec{u}(x, y) = (y, -x)$.

9.2. Find the area of the plane region bounded by the closed curve

$$x = \sin^2 \varphi, \quad y = \cos^3 \varphi.$$

Here φ is a parameter that ranges from 0 to 2π.

9.3. Let P be the rectangular parallelepiped in $\mathbb{R}^3$ given by the following inequalities:

$$0 \le x \le a, \quad 0 \le y \le b, \quad 0 \le z \le c.$$

Evaluate the integrals

$$\iiint_P \operatorname{div}(\vec{u})\, dV, \quad \iint_{\partial P} \vec{u} \cdot d\vec{S},$$

where $\vec{u}(x, y, z) = (x, y, z)$, by direct computation. Verify the Gauss theorem for the given P and $\vec{u}$.

9.4. Evaluate the following integrals over the sphere $x^2 + y^2 + z^2 = 1$:

(a) $\displaystyle\iint_S (x^3, y, z) \cdot d\vec{S}$; (b) $\displaystyle\iint_S (e^{xyz} y, -e^{xyz} x, 0) \cdot d\vec{S}$.

9.5. ($\star$) Let U be a region in $\mathbb{R}^3$ with a smooth boundary. We say that U is *simply connected* if for every non-self-intersecting loop (i.e. closed curve) γ in U there is a surface S in U such that $\gamma = \partial S$. Give a geometric argument that shows that for every smooth vector field $\vec{v}$ on a simply connected region U such that $\nabla \times \vec{v} = \vec{0}$, there exists a function φ on U such that $\vec{v} = \nabla\varphi$. The function φ is called the *potential* of $\vec{v}$ on U.

Hint: choose a base point $x_0 \in U$ and define

$$\varphi(x) = \int_\gamma \vec{u} \cdot d\vec{r}$$

for every point $x \in U$, where γ is some curve connecting x_0 to x. Use Stokes' theorem to show that φ is well-defined.