Complex numbers

Practice list

1. Compute
   a) $(-i)^2$; b) i^{10}; c) $(1+i)^{10}$; d) $(1-i)^{10}$; e) $(1+i)^{101}$.
2. Find $z, w \in \mathbb{C}$, such that $z + w = zw = 2$.
3. Find $1/(a + bi)$.
4. Compute the sum $1 + i + i^2 + i^3 + \ldots + i^{100}$.
5. Solve the following equations:
   a) $z^2 = 2$; b) $z^2 = -2$; c) $z^2 = 2i$; d) $z^2 = 1 + i$;
   e) $z^2 + 2z + 2 = 0$; f) $z^2 + (2i - 7)z + 13 - i = 0$.
6. Compute real and imaginary parts of $(a + bi)(c + di)$.
7. Prove that $z + \overline{z} \in \mathbb{R}$.
8. Prove that
   a) $\overline{z + w} = \overline{z} + \overline{w}$; b) $\overline{zw} = \overline{z}\,\overline{w}$; c) $\overline{z/w} = \overline{z}/\overline{w}$.
9. Compute
   a) $(1 + i)/(1 - i)$; b) $(8 + i)/(1 + 2i)$; c) $(5 + i)(7 - 6i)/(3 + i)$;
   d) $(1 + i)^5/(1 - i)^3$; e) $(1 + 3i)(8 - i)/(2 + i)^2$.
10. Show that if $zw \in \mathbb{R}$ and $z+w \in \mathbb{R}$ then either both z and w are real, or $z = \overline{w}$.
11. Show that $|z|^2 = z\overline{z}$.
12. Find the distance from z to w.
13. Show that $|zw| = |z||w|$.
14. Find the real and imaginary part of a complex number with absolute value r and argument φ.
15. Find the locus
    a) $z + \overline{z} = 1$; b) $z\overline{z} = 1$; c) $|z| = 1$; d) $|z - 1| = 1$;
    e) $|z| = |z+1|$; f) $|2z|^2 = z\overline{z}$; g) $\Re e\, z/(z - 1) = 0$; h) $|z-a| = |z-b|$.
16. Prove that the points 0, $1/z$ and $\overline{z}$ are collinear.
17. Show that the locus $z + \overline{z} = z\overline{z}$ is a circle. Find its center and radius.
18. Find all the $z \in \mathbb{C}$, such that $|z - 3| \le 2$ and $|z + 4i| \le 3$.
19. Show that the points z, w, $1/\overline{z}$, $1/\overline{w}$ belong to one circle.
20. Draw the locus $z\overline{z} + az + \overline{az} + c = 0$, where $a \in \mathbb{C}$, $c \in \mathbb{R}$. How does the answer depend on a and c?
21. Solve the following equations:
    a) $|z| - z = 1 + 2i$; b) $|z| + z = 2 + i$.

22. Show that

$(\cos\alpha + i\sin\alpha)^k = \cos k\alpha + i\sin k\alpha.$

23. Compute $(1 + \cos\alpha + i\sin\alpha)^k$.

24. Show that if $z + 1/z = 2\cos\alpha$, then $z^m + 1/z^m = 2\cos m\alpha$.

25. Write down the following complex numbers in the form $a + bi$:

a) $\sqrt{8\sqrt{3}i - 8}$; b) $\sqrt[3]{(27 - 54i)/(2 + i)}$; c) $\sqrt[3]{1 + i}$.

26. Find an integer n such that $(\frac{2+i}{2-i})^n = 1$.

27. Compute
a) $\cos x + \cos 2x + \cdots + \cos nx$;
b) $\sin x + \sin 2x + \cdots + \sin nx$;
c) $\sin^2 x + \sin^2 3x + \cdots + \sin^2(2n-1)x$;
d) $\cos x + 2\cos 2x + 3\cos 3x + \cdots + n\cos nx$.

28. Solve the following equations:

a) $x^4 + 6x^3 + 9x^2 + 100 = 0$; b) $x^4 + 2x^2 - 24x + 72 = 0$.

29. Compute

a) $1 + 2\varepsilon + 3\varepsilon^2 + \cdots + n\varepsilon^{n-1}$, where $\varepsilon^n = 1$;
b) $\varepsilon_0^k + \varepsilon_1^k + \cdots + \varepsilon_{n-1}^k$ ($\varepsilon_0, \ldots, \varepsilon_{n-1}$ — n-roots of unity).