Hyperbolic geometry-1

Upper half-plane model

Practice list

Consider the upper half-plane $\mathbb{H}_+ = \{z \,|\, \mathrm{Im}\, z > 0\}$. We look at $\mathbb{H}_+$ as a model of the hyperbolic plane. First, we give $\mathbb{H}_+$ the metric

$$ds = \frac{|dz|}{\mathrm{Im}\, z}.$$

The lines (geodesics) are vertical rays and semicircles orthogonal to $\partial\mathbb{H}_+$. The angles are Euclidean angles. From now on "the plane" means the hyperbolic plane $\mathbb{H}_+$. The boundary $\partial\mathbb{H}_+ = \mathbb{R} \cup \{\infty\}$ is called *the absolute.*

1. How many non-intersecting half-planes can you draw on $\mathbb{H}_+$?
2. Which lines you would say to be parallel?
3. Consider a line l and a point $z \notin l$. How many lines parallel to l contain z?
4. a) Show that transformations $z \to -\overline{z} + 2a$ and $z \to \frac{r^2}{\overline{z}-a} + a, \quad a, r \in \mathbb{R}, r > 0$, preserve $\mathbb{H}_+$. What is the geometric meaning of these maps?

   b)* Show that these transformations preserve the metric form on $\mathbb{H}_+$.

The transformations from Ex. 4 are called *reflections.* We proved that any reflection is an isometry of $\mathbb{H}_+$.

5. Prove that any transformation of the form $\frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{R}$, $ad - bc > 0$, and $\frac{a\overline{z}+b}{c\overline{z}+d}, \quad a, b, c, d \in \mathbb{R}$, $ad - bc < 0$ is a product of several reflections.

Thus, these transformations are isometries of $\mathbb{H}_+$. Möbius transformations are orientation-preserving isometries. Transformations of the form $\frac{a\overline{z}+b}{c\overline{z}+d}$ are orientation-reversing isometries.

6. Find an isometry that maps

   a) an arbitrary point $z \in \mathbb{H}_+$ to $w \in \mathbb{H}_+$;

   b) an arbitrary line to another fixed line;

   c) a triple of points of $\partial\mathbb{H}_+$ to $(0, 1, \infty)$.
7. How many reflections you need to map any triple of points of $\partial\mathbb{H}_+$ to another triple?
8. a) Show that if an isometry fixes all points of absolute then it is identity map.

   b) Show that if an isometry fixes three points of absolute then it is identity map.
9. Show that any orientation-preserving isometry is a Möbius map.
10. Find the distance $d(z_1, z_2)$ between two points $z_1 = x + iy_1$ and $z_2 = x + iy_2$.
11. Check the distance formula

    $$\cosh d(z, w) = 1 + \frac{|z-w|^2}{2\mathrm{Im}\,(z)\mathrm{Im}\,(w)},$$

    where $\cosh t = \frac{1}{2}(e^t + e^{-t})$ is the hyperbolic cosine.