Inversion

Practice list

For a circle C in the Euclidean plane the *inversion* i_C in C is the unique map from the complement of C to itself that fixes any point of C, exchanges the interior and exterior of C and takes circles orthogonal to C to themselves.

1. (a) Show the following fact from Euclidean geometry: If A is a point outside a circle C and l is a line through A intersecting C at P and P', the product $AP \cdot AP'$ is independent of l and equals AT^2, where AT is a segment tangent to C at T.

   (b) Show that if a circle C has center O and radius r, and a circle C_1 is orthogonal to C and contains a point P, then C_1 contains a point P' on the ray $\overline{OP}$ such that $OP \cdot OP' = r^2$.

   (c) Show that inversion is well-defined by proving that $i_C(P) = P'$.

2. (a) Denote by F_t the family of circles through $x \in C$ tangent to C. Show that $i_C(F_t) = F_t$. (Hint: consider the family F_o of circles through x orthogonal to C.)

   (b) Denote by T_λ the homothety with coefficient λ centered at the center of C. Show that $i_C \circ T_\lambda = T_{1/\lambda} \circ i_C$.

   (c) If O is the center of C, the inversion i_C takes

   - circles through O to lines;
   - rest circles to circles;
   - lines through O to themselves;
   - rest lines to circles through O.

3. Let C be the circle $x^2 + y^2 = 1$. Draw the image $i_C(P)$, where P is

   (a) the circle $(x-4)^2 + y^2 = 4$;

   (b) the circle $(x-\sqrt{2})^2 + (y-\sqrt{2})^2 = 1$;

   (c) the line $x = \frac{1}{2}$;

   (d) the square with vertices $(-1,1), (1,-1), (-1,-3), (-3,-1)$;

   (e) the upper half-plane $y \geq 0$.

4. Let i_C be the inversion in the circle C. Let C_1 and C_2 be a pair of tangent circles. Find the image $i_C(C_1 \bigcup C_2)$. How does it depend on C?

5. Show that i_C is conformal, i.e. it preserves angles.

6. Show that for any pair of non-intersecting circles C_1 and C_2 there exists a circle C such that i_C takes C_1 and C_2 to concentric circles.