Möbius transformations

Practice list

1. What is the geometric meaning of the transformation

   (a) $z \to az$, $a \in \mathbb{C}$, $|a| = 1$; (b) $z \to az$, $a \in \mathbb{R}$;

   (c) $z \to az$, $a \in \mathbb{C}$; (d) $z \to z + a$, $a \in \mathbb{C}$.

2. What is the geometric meaning of the transformation $w : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, $w(z) = \frac{1}{z}$?

   A *Möbius map* is a map which can be written in the form $g(z) = \frac{az+b}{cz+d}$, $ad - bc = 1$.

   This map acts on $\overline{\mathbb{C}}$ with the natural definition $g(\infty) = a/c$ and $g(-d/c) = \infty$ when $c \neq 0$, and $g(\infty) = \infty$ when $c = 0$.

3. (a) Show that the set of Möbius maps is a group;

   (b) show that any Möbius map can be represented as a composition of transformations $z \to \frac{1}{z}$, $z \to z + a$ and $z \to bz$ for some $a, b \in \mathbb{C}$.

   (c) show that any Möbius transformation takes lines and circles to lines and circles;

   (d) show that Möbius maps preserve angles.

4. Let (u_1, u_2, u_3) and (v_1, v_2, v_3) be triples of points of $\overline{\mathbb{C}}$. Find a Möbius map g, such that $g(u_k) = v_k$ for $k = 1, 2, 3$.

5. Show that if a Möbius map fixes at least three non-collinear points then it is the identity map.

6. (a) Find all Möbius maps that preserve the real axis;

   (b) Find all Möbius maps that preserve the disk $|z| \leq 1$.

7. Show that Möbius transformations preserve **cross ratio**

   $$[z_1, z_2, z_3, z_4] = \frac{z_3 - z_1}{z_3 - z_2} : \frac{z_4 - z_1}{z_4 - z_2}$$

   of four points.

8. Find all Möbius maps that take the upper half-plane $Im\ z > 0$ to the unit disk $|z| \leq 1$.