Jacobs University
School of Engineering and Science

Fall Term 2010

## Complex Analysis, Homework 1

**1.1.** A real manifold is *oriented* if for some atlas the Jacobian of any transition map is positive. Show that any Riemann surface is an oriented real manifold.

**1.2.** Show that the exponential map $\exp : \mathbb{C} \to \mathbb{C}^*$ is a covering. Prove that $\mathbb{C}^*$ is isomorphic (as Riemann surface) to the quotient $\mathbb{C}/L$, where $L = 2\pi i\mathbb{Z}$.

**1.3.** Prove the *Maximum Principle*: if $f$ is a holomorphic non-constant function on a connected open set $U$ of a Riemann surface $X$, then $f$ has no maximum in $U$.

**1.4.** Let $L$ be a lattice in $\mathbb{C}$, and $\lambda \in \mathbb{C}^*$.

(a) Show that the map $\varphi : \mathbb{C}/L \to \mathbb{C}/(\lambda L)$ assigning to every class $z + L$ the class $\lambda z + \lambda L$ is an isomorphism.

(b) Show that every complex torus is isomorphic to $\mathbb{C}/L$ for $L = \mathbb{Z} + \mathbb{Z}\tau$, where $|\mathrm{Re}\,\tau| \leq 1/2$, $\mathrm{Im}\,\tau > 0$, and $|\tau| \geq 1$ (the domain $D$ defined by these three inequalities is called *modular figure*).

(c) The group $SL_2(\mathbb{Z})$ of integer $2 \times 2$ matrices with unit determinant acts on the upper half-plane by linear-fractional transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$$

Let $L$ be a lattice, and $\gamma \in SL_2(\mathbb{Z})$. Show that $\mathbb{C}/L$ coincides with $\mathbb{C}/\gamma L$.

(d)($\star$) Show that if $z$ is an interior point of $D$, and $\gamma \in SL_2(\mathbb{Z})$, then $gz \notin D$.

(e)($\star$) Show that $D$ is the *fundamental domain* for the action of $SL_2(\mathbb{Z})$ on $\mathbb{H}$, i.e.

$$\bigcup_{\gamma \in SL_2(\mathbb{Z})} \gamma D = \mathbb{H}, \text{ and } D \cap \gamma D \text{ has no interior points for any } \gamma \in SL_2(\mathbb{Z}).$$

**1.5.** Show that a holomorphic map $f : X \to Y$ of compact Riemann surfaces is biholomorphic if and only if it has degree one.

**1.6.** Find singular points of the following curves in $\mathbb{C}P^2$:

(a) $y^2 z = x^3$; (b) $y^2 z^{n-2} = \prod_{i=1}^{n} (x - a_i z)$, $n \geq 4$.

**1.7.** Let $X$ be a smooth projective curve defined by equation $F(x, y, z) = 0$, where $F$ is a homogeneous polynomial of degree $d$. The number $d$ is called the *degree* of $X$. Define a projection $\pi : \mathbb{C}P^2 \setminus (0 : 0 : 1) \to \mathbb{C}P^1$, $\pi(x : y : z) = (x : y)$.

(a) What is the preimage $\pi^{-1}(z)$ for arbitrary $z \in \mathbb{C}P^1$?

(b) Show that the restriction $\pi : X \to \mathbb{C}P^1$ is a (ramified) covering of degree $d$.

**1.8.** Let $\varphi : f_1 \to f_2$ be a morphism of (ramified) coverings $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$, and let $x_2 \in X_2, x_1 \in \varphi^{-1}(x_2)$. Let $k_2$ be the ramification index of $f_2$ at $x_2$, and $k_1$ be the ramification index of $f_1$ at $x_1$. Show that $k_2 \mid k_1$.

**1.9.** Show that three Riemann surfaces $\mathbb{H}$, $\mathbb{C}$ and $\widehat{\mathbb{C}}$ are mutually non-isomorphic.