Jacobs University
School of Engineering and Science

Fall Term 2010

Complex Analysis, Homework 2

2.1. Let $f_i : X_i \to Y_i$, $i = 1, 2$, be ramified coverings, and $\psi : f_1 \to f_2$ is a morphism of coverings. Let $x_2 \in X_2$, $x_1 \in \psi^{-1}(x_2)$. Show that $\mathrm{mult}_{x_2}(f_2)$ divides $\mathrm{mult}_{x_1}(f_1)$.

2.2. ($\star$) Prove uniqueness of topological uniformization (up to isomorphism).

2.3. Let X be a Riemann surface, $\Gamma \subset \mathrm{Aut}(X)$ is discrete, and $\Gamma' \subset \Gamma$. Show that the natural map $X/\Gamma' \to X/\Gamma$ is a covering of Riemann surfaces.

2.4. The *Fermat curve* of degree d is given by equation $x^d + y^d + z^d = 0$ in $\mathbb{P}^2$.

(a) Show that Fermat curve is smooth.

(b) Show that the map f from the Fermat curve to $\mathbb{P}^1$ defined by $f : (x : y : z) \to (x : y)$ is a holomorphic map of degree d.

(c) Find all ramification and branch points of f.

(d) Use Riemann-Hurwitz formula to find the genus of the Fermat curve.

(e) Let $d = 2$. Show that the Fermat curve is isomorphic to $\mathbb{P}^1$.

2.5. The *Klein curve* X is defined by $xy^3 + yz^3 + zx^3 = 0$ in $\mathbb{P}^2$. Using the fact $g(X) = 3$, show that $|\mathrm{Aut}(X)| = 84(g - 1)$.

2.6. (a) A *line* in $\mathbb{P}^2$ is a curve of degree 1. Show that any line is smooth and isomorphic to $\mathbb{P}^1$.

(b) A *conic* in $\mathbb{P}^2$ is a curve of degree 2. Changing coordinates, show that every smooth conic is isomorphic to the Fermat curve of degree 2 (and thus, to $\mathbb{P}^1$).

2.7. Let $L \subset \mathbb{C}$ be a lattice of rank 2. Show that L contains at most 6 roots of unity.