Jacobs University
School of Engineering and Science

Fall Term 2010

Complex Analysis, Homework 3

Introduction to hyperbolic geometry

1. Upper half-plane model.

Consider the upper half-plane $\mathbb{H} = \{z \,|\, \mathrm{Im}(z) > 0\}$. We look at $\mathbb{H}$ as a model of the hyperbolic plane. First, we give $\mathbb{H}$ the metric

$$ds = \frac{|dz|}{\mathrm{Im}(z)}.$$

The lines (geodesics) are vertical rays and semicircles orthogonal to $\partial\mathbb{H}$. The angles are Euclidean angles. From now on "the plane" means the hyperbolic plane $\mathbb{H}$. The boundary $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$ is called *the absolute*.

3.1. How many non-intersecting half-planes can you draw on $\mathbb{H}$?

3.2. (a) Show that transformations $z \to -\overline{z} + 2a$ and $z \to \frac{r^2}{\overline{z}-a} + a, \quad a, r \in \mathbb{R}, r > 0$, preserve $\mathbb{H}$ and lines in $\mathbb{H}$. What is the geometric meaning of these maps?

(b) Show that these transformations preserve the metric form on $\mathbb{H}$.

The transformations above are called *reflections*. We proved that any reflection is an isometry of $\mathbb{H}$.

3.3. Prove that any transformation of the form $\frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{R}, ad - bc > 0$, and $\frac{a\overline{z}+b}{c\overline{z}+d}, \quad a, b, c, d \in \mathbb{R}, ad - bc < 0$ is a product of several reflections.

Thus, these transformations are isometries of $\mathbb{H}$. Möbius transformations are orientation-preserving isometries. Transformations of the form $\frac{a\overline{z}+b}{c\overline{z}+d}$ are orientation-reversing isometries.

3.4. Find an isometry that maps

(a) an arbitrary point $z \in \mathbb{H}$ to $w \in \mathbb{H}$;

(b) an arbitrary line to another fixed line;

(c) a triple of points of $\partial\mathbb{H}$ to $(0, 1, \infty)$.

3.5. How many reflections you need to map any triple of points of $\partial\mathbb{H}$ to another triple?

3.6. a) Show that if an isometry fixes all points of absolute then it is identity map.

b) Show that if an isometry fixes three points of absolute then it is identity map.

3.7. Show that any orientation-preserving isometry is a Möbius map.

3.8. Given $k \in \mathbb{R} \setminus 0$, show that the distance from any point of the curve $y = kx$ to the line $x = 0$ is the same.

3.9. Find the distance $d(z_1, z_2)$ between two points $z_1 = x + iy_1$ and $z_2 = x + iy_2$.

3.10. Check the distance formula

$$\cosh d(z, w) = 1 + \frac{|z - w|^2}{2\mathrm{Im}(z)\mathrm{Im}(w)},$$

where $\cosh t = \frac{1}{2}(e^t + e^{-t})$ is the hyperbolic cosine.

2. Unit disk model and classification of isometries.

3.11. (a) Find any Möbius map γ that takes the unit disk Δ to the upper half-plane $\mathbb{H}$.

(b) Show that the pullback metric $\gamma^* g$ on Δ of the hyperbolic metric g on $\mathbb{H}$ is

$$ds = \frac{2|dz|}{1-|z|^2}$$

(c) Compute the isometry group of Δ via conjugating the isometry group of $\mathbb{H}$ by γ^{-1}.

(d) What are the lines in the unit disc model?

(e) Write down the distance formula in the unit disk model.

3.12. Show that any two lines having no common points in $\mathbb{H} \cup \partial\mathbb{H}$ have a common perpendicular.

3.13. Let $\gamma \in PSL_2(\mathbb{R})$. Show that

(a) $\operatorname{tr}^2(\gamma) < 4$ iff γ has a unique fixed point in $\mathbb{H}$;

(b) $\operatorname{tr}^2(\gamma) = 4$ iff γ has a unique fixed point in $\partial\mathbb{H}$;

(c) $\operatorname{tr}^2(\gamma) > 4$ iff γ has two fixed point in $\partial\mathbb{H}$.

The isometries above are called, respectively *elliptic, parabolic,* and *hyperbolic.*

3.14. Show that every $\gamma \in PSL_2(\mathbb{R})$ is a product of two reflections. Characterize all the three types of isometries in terms of (relative) positions of the two mirrors of reflections.

3.15. Let γ_1, γ_2 be two commuting orientation-preserving isometries of $\mathbb{H}$. Show that they are of the same type, and their fixed points coincide.

3. Area.

3.16. Show that the sum of angles of any triangle is less than π.

3.17. (a) Show that there exists a triangle with angles $(0,0,0)$. This triangle is called *ideal.* Prove that any two ideal triangles are congruent.

(b) Show that any two triangles with angles $(0,0,\varphi)$ are congruent.

3.18. Denote by t the area of an ideal triangle. Let $f(\varphi)$ be the area of a triangle with angles $(\pi-\varphi, 0, 0)$.

(a) Prove that $f(\varphi) + f(\psi) = f(\varphi+\psi)$. Conclude that $f(\varphi) = \frac{\varphi}{\pi} t$.

(b) Show that $S_{ABC} = \frac{\pi - (\angle A + \angle B + \angle C)}{\pi} t$.

Hint: cut an ideal triangle into triangle ABC and triangles with angles $(0,0,\pi-\angle A)$, $(0,0,\pi-\angle B)$ and $(0,0,\pi-\angle C)$.

3.19. Write down the area of an ideal triangle as a double integral and compute it.

Thus, we proved the formula

$$S_{ABC} = \pi - (\angle A + \angle B + \angle C)$$

3.20. (a) What can you say about a sum of angles of a polygon?

(b) Let P be an acute-angled polygon. Let l and m be the lines containing two disjoint sides of P. Show that l and m have no common points.

3.21. Show that if two triangles have the same angles, then they are congruent.

4. Hyperbolic surfaces.

3.22. Let $x \in \partial\mathbb{H} \cup \mathbb{H}$, and $\Gamma \subset PSL_2(\mathbb{R})$ is discrete. Show that

(a) Γ_x is abelian;

(b) Γ_x is cyclic;

(c) Γ contains a parabolic element, then $\mathbb{H}/\Gamma$ is not compact;

(d) if $\mathbb{H}/\Gamma$ is compact and Γ acts without fixed points, then Γ contains hyperbolic elements only (Γ is *purely hyperbolic*);

(e) if $\mathbb{H}/\Gamma$ is compact then the only element commuting with all the Γ is the trivial one;

(f) Γ contains no subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

3.23. Let $\gamma \in \Gamma$ be hyperbolic, $\Gamma \subset PSL_2(\mathbb{R})$ is discrete. Show that the projection of the unique invariant line of γ is a closed geodesic in $\mathbb{H}/\Gamma$.

3.24. Let $X = \mathbb{H}/\Gamma$ be compact of genus g, Γ acts without fixed points.

(a) Show that there exists $2g$ closed geodesics, such that cutting X along them results in a polygon. (*Hint*: consider free homotopy classes of loops on X).

(b) Show that the area of X is equal to $-2\pi\chi(X)$.