Jacobs University
School of Engineering and Science

Fall Term 2010

Complex Analysis, Homework 4

4.1. Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a meromorphic function defined by

$$f(z) = \frac{4(z^2 - z + 1)^3}{27z^2(z-1)^2}$$

(a) Show that f defines a Galois covering $\hat{\mathbb{C}} \to \hat{\mathbb{C}}/\Gamma$, where Γ is isomorphic to symmetric group S_3 and generated by automorphisms of $\hat{\mathbb{C}}$ $\quad z \to 1 - z$ and $z \to 1/z$.

(b) Solving the equation $f'(z) = 0$, find the ramification points.

(c) Compute the ramification divisor R_f.

4.2. Let X be a compact Riemann surface, and suppose that there exists a meromorphic function f on X having precisely two simple poles. Show that X is hyperelliptic.

4.3. Let $P(z)/Q(z)$ be a meromorphic function on $\hat{\mathbb{C}}$, $\deg P \leq \deg Q - 2$, and Q has no multiple roots. Show that

$$\sum_{a\colon Q(a)=0} \frac{P(a)}{Q'(a)} = 0$$

4.4. Let $f : X \to Y$ be a Galois covering with Galois group Γ (i.e. $Y = X/\Gamma$). Show that $f^*(\Omega_H(Y))$ coincides with the set of holomorphic 1-forms on X that are invariant under Γ.

4.5. Compute the canonical map for the Fermat curve $x^4 + y^4 + z^4 = 0$.

4.6. Let X be a hyperelliptic curve $y^2 = \prod_{i=1}^{n} (x - a_i)$. Consider x and y as meromorphic functions on X. Compute the proncipal divisors (x) and (y). Does the result depend on the parity of n?

4.7. (a) Let X be an elliptic curve. Show that canonical divisors are exactly principal divisors.

(b) Let X be a projective curve defined by $y^2 z = x^3 - xz^2$. Compute the intersection divisors for coordinate hyperplanes $x = 0$, $y = 0$, and $z = 0$.

(c) Show that two divisors on $\hat{\mathbb{C}}$ are linearly equivalent if and only if they have the same degree.