Jacobs University
School of Engineering and Science

Fall Term 2010
04.12.10

Complex Analysis, Final exam

Three of 7 problems are bonus ones.

In all the problems X, Y denote compact Riemann surfaces, $g(X)$ is the genus of X.

F.1. (a) Let $f : \mathbb{P}^1 \to Y$ be a non-constant holomorphic map. Show that Y is isomorphic to $\mathbb{P}^1$.

(b) Let $f : X \to Y$ be a non-constant holomorphic map. Is it true that $g(X) \geq g(Y)$?

F.2. Compute the number of mutually non-isomorphic non-ramified coverings of degree 4 of a given elliptic curve.

F.3. (a) Show that diameter of a (hyperbolic) circle inscribed into a hyperbolic triangle does not exceed $\ln 3$.

(b) Is it possible to tile a regular hyperbolic triangle with side of length 100 by copies of a regular hyperbolic triangle with side of length 1?

F.4. Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a meromorphic function defined by

$$f(z) = \frac{z(z-2)}{z^2 - z + 1}$$

(a) Show that f defines a Galois covering $\hat{\mathbb{C}} \to \hat{\mathbb{C}}/\Gamma$, and find Γ explicitly.

(b) Find the ramification points of f and compute the ramification divisor R_f.

F.5. Denote by X_d a Fermat curve of degree d in $\mathbb{P}^2$, i.e. X_d is given by equation

$$x^d + y^d + z^d = 0$$

(a) Show that X_4 is not hyperelliptic.

(b) Show that X_d is not hyperelliptic for any $d \geq 4$.

(c) Is it true that any smooth projective curve of degree at least 4 in $\mathbb{P}^2$ is not hyperelliptic?

F.6. Define hyperelliptic curves X_1 and X_2 in $\mathbb{P}^2$ by equations

$$y^2 = x^3 - 2x + 1 \qquad \text{and} \qquad y^2 = x^{2011} - 1$$

respectively. Denote by φ_i the covering $X_i \to \hat{\mathbb{C}}$, $\varphi_i(x, y) = x$.

Let $\omega = y^3 dy$ be a meromorphic 1-form on X_1. Show that there exists a meromorphic 1-form $\tilde{\omega}$ on $\hat{\mathbb{C}}$ such that $\omega = \varphi_1^* \tilde{\omega}$. Compute the divisor $(\varphi_2^* \tilde{\omega})$ of $\varphi_2^* \tilde{\omega}$.

F.7. Show that on every Riemann surface of genus 3 there exists at least one meromorphic function of degree

(a) at most 4;

(b) at most 3.