Jacobs University
School of Engineering and Science

Fall Term 2010
29.10.10

Complex Analysis, Midterm problems

Due Date: Monday, November 1, 17:00.

Problems marked $(\star)$ are bonus ones.

M.1. Let G be a finite group which acts effectively on a compact Riemann surface X (i.e., the action of any element $g \in G$ is not the identity map).

(a) Show that if G acts by automorphisms of X then there are only finitely many points with a nontrivial stabilizer.

(b) Show that if G acts by automorphisms of X then the quotient X/G is homeomorphic to a smooth connected compact real manifold.

(c) Provide an example of a $\mathbb{Z}_2$-action on $\mathbb{P}^1$ such the the quotient $\mathbb{P}^1/G$ is not homeomorphic to a smooth real manifold.

(d) Provide an example of a $\mathbb{Z}_2$-action on $\mathbb{P}^1$ such the the quotient $\mathbb{P}^1/G$ is homeomorphic to a smooth real manifold but has no complex structure.

M.2. Show that $\mathbb{C} \setminus \{0, 1, 2\}$ and $\mathbb{C} \setminus \{0, 1, 3\}$ are not biholomorphically equivalent.

M.3. Compute genus of the following curves in $\mathbb{P}^2$:

(a) $z^4 - y^4 + x^4 - 4xy^2z = 0$;

(b) $y^2z^{n-2} = \prod_{i=1}^{n}(x - a_iz)$, $n \geq 4$, $a_i \neq a_j$ for $i \neq j$.

M.4. Let X be a quotient of $\mathbb{C}$ by a lattice generated by $1, \tau \in \mathbb{C}$. Prove that there exists a nonconstant meromorphic function on X.

M.5. Prove or disprove: any automorphism of a compact Riemann surface of genus $g > 1$ has a fixed point.

M.6. (a) Show that hyperbolic circles in the upper halfplane and the unit disk models are Euclidean circles.

(b) Write down the equation of a hyperbolic circle (in the upper halfplane model) of radius $\log 2$ centered at i.

M.7. $(\star)$ Construct a compact Riemann surface of genus g having an automorphism group of order $84(g - 1)$.