Jacobs University
School of Engineering and Science

Fall Term 2010

Discrete Mathematics, Homework 1

Due Date: Thursday, September 30, in class.

Problems marked ($\star$) are bonus ones.

1.1. Compute the number of 7-digit phone numbers, such that

(a) they contain at least four sevens;

(b) they contain at least two nines and two sevens;

(c) each digit is less than the preceding one;

(d) each digit does not exceed the preceding one.

1.2. ($\star$) Amongst 7-digit phone numbers, choose the set A containing no pair of sevens in a row, and the set B containing no eight after seven. Which set is larger? Compute $|A| - |B|$.

1.3. Which lines of Pascal triangle contain odd numbers only?

1.4. Let the integers from 1 to n be arranged in some order around a circle, and let k be an integer, $1 \leq k \leq n$. Show that there are k adjacent numbers whose sum is at least $k(n+1)/2$.

1.5. Fifty identical hamburgers are divided amongst twenty people. How many ways to do that, if

(a) none is left over?

(b) none is left over, and every persom received at least one?

(c) some hamburgers may left over?

1.6. Unlimited number of jelly beans of five colors is available.

(a) How many ways are there to select 25 jelly beans?

(b) How many ways are there to select 25 jelly beans if we must take at least two of each color?

1.7. Observe that $\binom{n-m}{r-m}$ is the number of r-subsets of an n-set that contain a fixed m-set M. Using inclusion-exclusion principle, deduce that

$$\binom{n-m}{r-m} = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \binom{n-k}{r}$$

1.8. Prove the following identities for generating functions:

(a) $\sin^2(x) + \cos^2(x) = 1$; (b) $(1+x)^\alpha (1+x)^\beta = (1+x)^{\alpha+\beta}$; (c) $\exp(\log(1-x)^{-1}) = (1-x)^{-1}$;

(d) $\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots + \frac{(-1)^{n+1}}{n} x^n + \ldots$; (e) $\log(1-x)^{-\alpha} = \alpha \log(1-s)^{-1}$

1.9. Show that for any real α and positive integer m:

(a) $\binom{\alpha}{m} = (-1)^m \binom{m-\alpha-1}{m}$; (b) $\binom{\alpha}{m} = \binom{\alpha-1}{m} + \binom{\alpha-1}{m-1}$; (c) $\sum_{m=0}^{n} \binom{\alpha+m}{m} = \binom{\alpha+n+1}{n}$.

1.10. Compute generating functions for the following sequences:

(a) $1, q, q^2, q^3, \ldots$; (b) $1, 2, 3, 4, 5, 6, \ldots$.

1.11. (a) For which n the Fibonacci number F_n is even?

(b) Show that $F_{n+1} = \sum_{k \le n-k} \binom{n-k}{k}$.

(c) Show that $F_n^2 + F_{n+1}^2 = F_{2n+1}$.

1.12. Show that the following definitions of Catalan numbers are equivalent:

(a) $C_0 = 1$, $C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$;

(b) $C_0 = 1$, C_n is the number of triangulations of a convex polygon with $n+2$ sides;

(c) $C_0 = 1$, C_n is the number of balanced bracket strings of length $2n$ (a string containing equal number of left and right brackets is *balanced* if for any starting interval the number of right brackets does not exceed the number of left brackets);

(d) C_n is the number of paths from the left bottom angle of an $n \times n$ board to the right upper one, going by the edges of cells to the right and up only, and not crossing the main diagonal.

1.13. ($\star$) Show that for any positive integer k there are infinitely many Fibonacci numbers F_n satisfying

$$F_n \equiv -1 \pmod{10^k}$$