Jacobs University
School of Engineering and Science

Fall Term 2010

Discrete Mathematics, Homework 2

Due Date: Wednesday, October 13.

Problems marked ($\star$) are bonus ones.

2.1. Let $B(x)$ be a generating function, $B(0) = 0, B'(0) \neq 0$. Show that there exist generating functions $A(x)$, $A(0) = 0$ and $C(x)$, $C(0) = 0$ such that $B(A(x)) = C(B(x)) = x$. Is it true that $A = C$?

2.2. Let $A(x)$ and $B(x)$ be generating functions, $A(0) \neq 0$, $B(0) = 0$, $B'(0) \neq 0$. Suppose that A and B satisfy *Lagrange equation*: $B(x) = xA(B(x))$. Prove *Lagrange Theorem*: given one of A and B, the other is uniquely determined.

2.3. Let C_n and F_n be n-th Catalan and Fibonacci number, p prime, k positive integer.

(a) Show that $p \mid C_{p^k-2}$.

(b) Show that $C_p \equiv 2 \pmod p$.

(c) ($\star$) Show that $C_{p^k} \equiv 2 \pmod p$.

(d) Show that $F_n \mid F_{2n}$.

2.4. Are the generating functions for the following sequences rational?

(a) $a_n = n^2$; (b) $b_n = 1/n^2$.

2.5. Consider paths on $\mathbb{R}_{\geq 0}$ starting at zero and consisting of segments of length one to the right or left. Find the generating function for the number of paths of length k if

(a) paths end at zero;

(b) paths end at $N > 0$.

2.6. Let $A(x)$ be a generating function for the sequence $\{a_n\}$. Find the generating function for the sequence

(a) $a_0 + a_1,\ a_1 + a_2,\ a_2 + a_3,\ \ldots$;

(b) $a_0,\ a_0 + a_1,\ a_0 + a_1 + a_2,\ a_0 + a_1 + a_2 + a_3,\ \ldots$

2.7. Show that the number of partitions of n into distinct summands is equal to the number of partitions of n into odd summands.

2.8. Show that

(a) $\prod\limits_{k\geq 1}(1 + x^k) = \prod\limits_{k\geq 1} \frac{1}{(1-x^{2k-1})}$;

(b) $\prod\limits_{k\geq 0}\sum\limits_{m=0}^{9}(x^{10^k m}) = \frac{1}{1-x}$ (use uniqueness of decimal notation of integers).

2.9. ($\star$) A 6-digit ticket is *lucky* if the sum of the first three digits is equal to the sum of the last three ones. Denote by M_{27} the set of tickets with sum of digits equal to 27.

(a) Show that the number of lucky tickets is equal to $|M_{27}|$.

Let M be the set of 6-tuples of non-negative integers (not obligatory digits!) $(n_1, \ldots, n_6)$ with sum equal to 27, i.e. $n_1 + \ldots + n_6 = 27$. Define property U_i, $1 \leq i \leq 6$ in the following way: $(n_1, \ldots, n_6) \in M$ has property U_i if $n_i \geq 10$. Let N_i be the number of elements of M with property U_i, N_{ij} be the set of elements of M with property U_i and U_j, etc.

(b) Show that $|M| = \binom{32}{5}$, and $N_i = \binom{22}{5}$ for any i.

(c) Show that $N_{ij} = \binom{12}{5}$, and $N_{ijk} = 0$ for any distinct i, j, k.

(d) Use inclusion-exclusion to show that the number of lucky tickets is equal to

$$\binom{32}{5} - 6\binom{22}{5} + 15\binom{12}{5}$$

2.10. ($\star$) (a) Find all the permutations $\tau \in S_n$ commuting with all permutations from S_n.

(b) Show that every permutation is a product of transpositions.

(c) A permutation $\tau \in S_n$ is *even* if τ is a product of even number of transpositions, and *odd* if τ is a product of odd number of transpositions. Show that parity is well defined.

(d) Show that the parity of $\tau \in S_n$ is equal to the parity of the number of pairs $(i < j)$ such that $\tau(i) > \tau(j)$.

(e) Show that the set of even permutations in S_n form a group (denoted by Alt_n). Show that $|\mathrm{Alt}_n| = n!/2$

(f) Show that S_n can be generated by two elements.

(g) Show that $|\mathrm{Alt}_n|$ is generated by cycles of length 3.

(h) Show that the Fifteen puzzle becomes unsolvable if one changes squares 14 and 15.

2.11. How many different necklaces having six beads can be formed using three different kinds of beads if we discount:

(a) Both reflections and rotations?

(b) Rotations only?

(c) Just one rotation?

2.12. The commander of a space cruiser wishes to post four sentry ships arrayed around the cruiser at the vertices of a tetrahedron for defensive purposes, since an attack can come from any direction.

(a) How many ways are there to deploy the ships if there are two different kinds of sentry ships available, and we discount all symmetries of the tetrahedral formation?

(b) How many ways are there if there are three different kinds of sentry ships available?

2.13. (a) How many ways are there to label the faces of a cube with the numbers 1 through 6 if each number may be used more than once.

(b) What if each number may only be used once?