Jacobs University
School of Engineering and Science

Fall Term 2010

Discrete Mathematics, Homework 3

Due Date: Thursday, October 28.

Problems marked ($\star$) are bonus ones.

3.1. What is the pattern inventory for coloring n objects using m colors if the group of symmetries is Alt_n?

3.2. Use Polya's enumeration formula to determine the number of non-equivalent colorings of the faces of a cube by the colors red, green and yellow, where red is used 3 times, green is used twice, and yellow is used only once.

3.3. Determine the pattern inventory for coloring the vertices of a regular p-gon, p being prime, with m colors if the group of symmetries consists of the rotations by multiples of the angle $2\pi/p$.

3.4. Use combinatorial arguments to determine simple formulas for $\left\{ n \atop 2 \right\}$ and $\left\{ n \atop n-2 \right\}$.

3.5. Let $r_{n,k}$ be the number of ways to divide n people into k groups, with at least two people in each group. Use a combinatorial argument to show that $r_{n,k}$ satis

es the recurrence relation $r_{n,k} = kr_{n-1,k} + (n-1)r_{n-2,k-1}$.

3.6. Define

$$F_n(x) = \sum_{k=0}^{n} \left\{ n \atop k \right\} x^{\underline{k}},$$

where $x^{\underline{k}} = x(x-1)\dots(x-(k-1))$. Show that

(a) $F_n(x) = \sum_{k=0}^{n} (-1)^{n-k} \left\{ n \atop k \right\} x^{\overline{k}}$;

(b) $F_n(x) = x^n$;

(c) $\sum_{k=0}^{n} \left\{ n \atop k \right\} \left[k \atop m \right] (-1)^{n-k} = 1$ if $n = m$, and 0 otherwise.

3.7. Is it true that a graph having exactly two vertices of odd degree must contain a path from one to the other? Give a proof or counterexample.

3.8. Let P_1 and P_2 be two paths of maximum length in a connected graph G. Prove that P_1 and P_2 have a common vertex.

3.9. Without computing the matrix directly, find A^4, where A is the adjacency matrix of $K_{2,3}$.

3.10. Let A be the adjacency matrix for the graph G.

(a) Show that the (j, j) entry of A^2 is the degree of v_j.

(b) Show that the number of triangles in G is the trace of A^3 divided by 6.

3.11. Let e be an edge of the complete graph K_n. Use Cayley's Theorem to prove that $K_n \setminus e$ has $(n-2)n^{n-3}$ spanning trees.

3.12. ($\star$) Use the Matrix Tree Theorem to prove Cayley's formula.

3.13. ($\star$) Find the number of spanning trees of the complete bipartite graph $K_{m,n}$.