Jacobs University
School of Engineering and Science

Fall Term 2010

Discrete Mathematics, Homework 5

Due Date: Thursday, November 25, in class.

Problems marked ($\star$) are bonus ones.

5.1. Prove that a graph of order at least two is bipartite if and only if it is 2-colorable.

5.2. A graph G is k-critical if $\chi(G) = k$ but $\chi(G \setminus v) < k$ for any $v \in V(G)$.

(a) Find all 1-critical and 2-critical graphs.

(b) Give an example of 3-critical graph.

(c) Show that any k-critical graph is connected.

(d) Show that if G is k-critical, then $\delta(G) \geq k - 1$.

(e)($\star$) Find all 3-critical graphs.

5.3. Show that if G is connected, not regular, not complete and not a cycle, then $\chi(G) \leq \Delta(G)$.

5.4. Show that if G is connected and has n vertices, then $\chi(G) \leq n + 1 - \alpha(G)$ (where $\alpha(G)$ is the independence number), and the bound is sharp.

5.5. Find the chromatic polynomials of the following graphs:

(a) $K_{1,5}$; (b) C_5; (c) K_4 without one edge; (d) K_5 without one edge.

5.6. Let G be a graph of order n, and let $c_G(k)$ be its chtomatic polynomial. Show that

(a) the leading coefficient of $c_G(k)$ is 1;

(b) the constant term is 0;

(c) the coefficients alternate in sign;

(d) the negative coefficient of the k^{n-1} term is the number of edges in G.

5.7. Show that $k^4 - 4k^3 + 3k^2$ is not a chtomatic polynomial for any graph.

5.8. Find the minimal size of maximal matching in

(a) C_{10}; (b) C_{11}; (c) C_n.

5.9. ($\star$) Let $\{S_1, \ldots, S_k\}$ be a family of finite sets. A *system of distinct representatives* (SDR) for $\{S_1, \ldots, S_k\}$ is a collection of distinct elements $\{x_1, \ldots, x_k\}$ such that $x_i \in S_i$, $1 \leq i \leq k$.

(a) Let $S_1 = \{1, 2\}$, $S_2 = \{2, 5\}$, $S_3 = \{2\}$, $S_4 = \{1, 2, 5\}$, $S_5 = \{2, 3, 5\}$. Find SDR for the collection $\{S_1, S_2, S_3, S_5\}$; show that the collection $\{S_1, S_2, S_3, S_4\}$ has no SDR.

(b) Use Hall's Theorem to prove that $\{S_1, \ldots, S_k\}$ has SDR if and only if for any $m \in [1..k]$ the union af any m of these sets contain at least m elements.
Hint: create a bipartite graph with $X = \{S_1, \ldots, S_k\}$ and $Y = S_1 \cup \cdots \cup S_k$.

5.10. ($\star$) Let G be a bipartite graph with partite sets X and Y. Denote by δ_X and Δ_Y the minimum and the maximum degrees of vertices in X and Y respectively. Show that if $\delta_X \geq \Delta_Y$, then there exists a matching of G that saturates X.

5.11. (a) Find a 2-coloring of the edges of the graph K_{13} that proves that $R(3, 5) \geq 14$. Justify your answer.

(b) Show that $R(3, 5) = 14$.

5.12. ($\star$) Prove that $R(4, 4) = 18$.

5.13. Find the graph Ramsey numbers $R(P_3, C_4)$, $R(C_4, C_4)$, and $R(K_{1,3}, K_{1,3})$.