Jacobs University
School of Engineering and Science

Fall Term 2010

Discrete Mathematics, Homework 6

Due Date: Thursday, December 2, in class.

Problems marked ($\star$) are bonus ones.

6.1. Prove, under ZFC, that $\bigcap X$ is unique.

6.2. The infinity axiom says that there is a set Z such that

(1) $\emptyset \in Z$; (2) if $x \in Z$, then $\{x\} \in Z$. Let Z be such a set and T be the set of all subsets of Z that satisfy the properties (1) and (2) of the infinity axiom. Define $Z_0 = \bigcap T$.

(a) Show formally that T and Z_0 do exist by using the ZFC axioms.

(b) Prove that if X is any set satisfying the properties (1) and (2) of the infinity axiom then, $Z_0 \subset X$. (In this sense, Z_0 is a minimal infinite set.)

6.3. (a) Show that there is no X such that $|P(X)| = \aleph_0$.

(b) Find the cardinality of the set of bijections of $\mathbb{N}$ onto itself.

(c)($\star$) Let X, Y be sets, and $|X \cup Y| = c$, where c is cardinality of continuum. Show that either $|X| = c$ or $|Y| = c$.

6.4. An element p of a partially ordered set (*poset*) P is *maximal* if there are no elements p' with $p < p'$, and *maximum* if $p' < p$ for any $p' \in P$.

(a) Show that if a maximum element exists then it is unique;

(b) show that a maximum element of P is a unique maximal element;

(c) provide an example of a poset containing a unique maximal element but no maximum element.

6.5. ($\star$) Show that under ZF the following are equivalent:

(a) Axiom of Choice.

(b) Zorn's Lemma: Let P be a poset such that for every linearly ordered subset (*chain*) M in P there is an element $m \in P$ such that $x \leq m$ for any $x \in M$ (m is called *an upper bound of* M). Then there exists a maximal element m_0 of P such that m_0 is an upper bound of M.

(c) Every set can be well-ordered.