Jacobs University
School of Engineering and Science

Fall Term 2011

ESM 1B, Homework 10

Due Date: 14:00 Wednesday, November 23.

Explain your answers! Problems marked ($\star$) are bonus ones.

10.1. Compute divergence and curl of the following vector fields:

(a) $\vec{a} = (yz, xz, xy)$;

(b) $\vec{a} = (x^2y^3, 0, xz^2)$;

(c) $\vec{a} = (\sin x, x\cos y, \sin z)$.

10.2. Let f, g be scalar fields, and let $\vec{u}$, $\vec{v}$ be vector fields. Prove the following formulas.

(a) $\operatorname{div}(f\vec{u}) = (\operatorname{grad} f)\cdot\vec{u} + f \operatorname{div}\vec{u}$;

(b) $\operatorname{curl}(f\vec{u}) = (\operatorname{grad} f)\times\vec{u} + f\operatorname{curl}\vec{u}$;

(c) $\operatorname{div}(\vec{u}\times\vec{v}) = \vec{v}\cdot(\operatorname{curl}\vec{u}) - \vec{u}\cdot(\operatorname{curl}\vec{v})$;

(d) $\operatorname{div}(\operatorname{grad} f \times \operatorname{grad} g) = 0$.

10.3. Find a function f and vector field $\vec{a}$ (or prove that it does not exist) such that

(a) $\operatorname{grad} \vec{f} = (y\cos x, x\cos y, xyz)$;

(b) $\operatorname{curl}\vec{a} = (xy, -yz, xy)$.

10.4. Compute components of $\operatorname{curl}\vec{a}$ in cylindrical coordinates, where $\vec{a} = (a_\rho, a_\varphi, a_z)$.

10.5. Let $(\rho, \vartheta, \varphi)$ be spherical coordinates:

$$x = \rho\sin\vartheta\cos\varphi, \quad y = \rho\sin\vartheta\sin\varphi, \quad z = \rho\cos\vartheta.$$

Let $\vec{e}_\rho$, $\vec{e}_\vartheta$ and $\vec{e}_\varphi$ be the unit vectors of the same direction as the partial derivatives

$$\frac{\partial\vec{r}}{\partial\rho}, \quad \frac{\partial\vec{r}}{\partial\vartheta}, \quad \frac{\partial\vec{r}}{\partial\varphi}.$$

Prove that the gradient of a scalar field f has the following expression in the spherical coordinates:

$$\nabla f = \frac{\partial f}{\partial\rho}\vec{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial\vartheta}\vec{e}_\vartheta + \frac{1}{\rho\sin\vartheta}\frac{\partial f}{\partial\varphi}\vec{e}_\varphi.$$

You may use the textbook but please give a detailed and complete computation.

10.6. ($\star$) Let $f = 1/r$, where $r = \sqrt{x^2+y^2+z^2}$. Write down the expression for vector field $\vec{u} = \operatorname{grad} f$

(a) in Cartesian coordinates;

(b) in cylindrical coordinates;

(c) in spherical coordinates.

(d) Compute $\operatorname{div}\vec{u}$ and $\operatorname{curl}\vec{u}$.