Jacobs University
School of Engineering and Science

Fall Term 2011

ESM 1B, Homework 11

Due Date: 14:00 Wednesday, November 30.

<u>Explain your answers!</u> Problems marked ($\star$) are bonus ones.

11.1. Find the area of the plane region bounded by the closed curve

$$x = \sin^3 \varphi, \quad y = \cos \varphi.$$

Here φ is a parameter that ranges from 0 to 2π.

11.2. Let P be the rectangular parallelepiped in $\mathbb{R}^3$ given by the following inequalities:

$$0 \leq x \leq a, \quad 0 \leq y \leq b, \quad 0 \leq z \leq c.$$

Evaluate the integrals

$$\iiint_P \operatorname{div}(\vec{u})\, dV, \quad \iint_{\partial P} \vec{u} \cdot d\vec{S},$$

where $\vec{u}(x, y, z) = (x, y, z)$, by direct computation. Verify the Gauss theorem for the given P and $\vec{u}$.

11.3. Evaluate the following integrals over the sphere $x^2 + y^2 + z^2 = 1$:

(a) $\iint_S (x^3, y, z) \cdot d\vec{S}$; (b) $\iint_S (e^{xyz} y, -e^{xyz} x, 0) \cdot d\vec{S}$.

11.4. ($\star$) Let U be a region in $\mathbb{R}^3$ with a smooth boundary. We say that U is *simply connected* if for every non-self-intersecting loop (i.e. closed curve) γ in U there is a surface S in U such that $\gamma = \partial S$. Give a geometric argument that shows that for every smooth vector field $\vec{v}$ on a simply connected region U such that $\nabla \times \vec{v} = \vec{0}$, there exists a function φ on U such that $\vec{v} = \nabla\varphi$. The function φ is called the *potential* of $\vec{v}$ on U.

Hint: choose a base point $x_0 \in U$ and define

$$\varphi(x) = \int_\gamma \vec{u} \cdot d\vec{r}$$

for every point $x \in U$, where γ is some curve connecting x_0 to x. Use Stokes' theorem to show that φ is well-defined.