Jacobs University
School of Engineering and Science

Fall Term 2011

ESM 1B, Homework 4

Due Date: 14:00 Wednesday, October 5.

Explain your answers! Problems marked ($\star$) are bonus ones.

4.1. *Cylindrical coordinates* (ρ, ϑ, ζ) are defined by the following formulas:

$$x = \rho \cos \vartheta, \quad y = \rho \sin \vartheta, \quad z = \zeta.$$

where (x, y, z) are the Cartesian coordinates. Suppose that $f(x, y, z)$ is a function, whose partial derivatives with respect to the Cartesian coordinates are known. Write a formula for the partial derivatives of f with respect to the cylindrical coordinates.

4.2. The equation

$$3y = z^3 + 3xz$$

defines z as an implicit function of x and y. Evaluate all the three second order partial derivatives of z with respect to x and y.

Verify that z is a solution of equation

$$x\frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial x^2} = 0$$

4.3. Using the chain rule, transform the equation

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial^2 \varphi}{\partial y^2}$$

to new coordinates $s = \frac{1}{2}(x + y)$, $t = \frac{1}{2}(x - y)$. Show that φ has the form

$$f(x + y) + g(x - y)$$

for some functions f and g of one variable.

4.4. Suppose that a function f satisfies the equation

$$yf_x + xf_y = 0.$$

Transform this equation to new coordinates $s = x^2 - y^2$, $t = 2xy$ to show that f is a function of $x^2 - y^2$ only.

4.5. Find the second-order approximation of the following function at $(2, 1)$:

$$f(x, y) = \exp(x^2 + y^2).$$

4.6. Find all stationary points of the function

$$f(x, y) = x^3 + 2x^2 - x - y^3 + 3y^2 + 3y + 6$$

Determine the nature of the stationary points (maxima, minima, saddle points or none of these).

4.7. Give an example of a function on $\mathbb{R}^2$ having

(a) a local minimum at $(0, 0)$ and a local maximum at $(1, 0)$;

(b)($\star$) a local maximum at $(0, 0)$ and local minima at $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$;

(c)($\star$) saddle points at $(0, 0)$, $(1, 1)$, $(-1, -1)$, $(-1, 1)$, and $(1, -1)$.