Jacobs University
School of Engineering and Science

Fall Term 2011

ESM 1B, Homework 8

Due Date: 14:00 Wednesday, November 2.

Explain your answers! Problems marked ($\star$) are bonus ones.

8.1. Compute the Jacobian $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ for the following changes of coordinates:

(a) $x = uv\cos w, \quad y = uv\sin w, \quad z = \frac{1}{2}(u^2 - v^2)$ (parabolic coordinates);

(b) $x = f(u+v), \quad y = g(u-v), \quad z = w.$

In (b), express the answer in terms of the derivatives of the functions f and g.

8.2. Find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

in terms of the parameters $a, b, c > 0$.
Hint: consider the following change of variables: $x = au, \quad y = bv, \quad z = cw.$

8.3. Let D be a 3-dimensional body. The *moment of inertia* of D (with respect to $(0,0,0)$) is defined as

$$\iiint_D R^2 \rho(x,y,z)\,dx\,dy\,dz,$$

where $R = \sqrt{x^2+y^2+z^2}$ is the distance of point (x,y,z) to $(0,0,0)$, and $\rho(x,y,z)$ is the density at point (x,y,z). Find the moment of inertia for the cylinder

$$x^2 + y^2 \leq a^2, \quad 0 \leq z \leq b$$

with uniform density ρ_0.

Hint: change to *cylindrical coordinates* (r, ϑ, ζ), where $x = r\cos\vartheta$, $y = r\sin\vartheta$, $z = \zeta$.

8.4. (a) Prove the Leibniz rule for differentiation of the cross-product:

$$\frac{d}{dt}(\vec{a}(t) \times \vec{b}(t)) = \left(\frac{d}{dt}\vec{a}(t)\right) \times \vec{b}(t) + \vec{a}(t) \times \left(\frac{d}{dt}\vec{b}(t)\right).$$

(b) Suppose that a particle moves with uniform speed. Show that the acceleration vector is perpendicular to the velocity vector.
Hint: use the equation $\vec{v}(t) \cdot \vec{v}(t) = const.$

8.5. ($\star$) Let $f : [a,b] \to \mathbb{R}$ be a real function. Find the volume of the 3-dimensional domain D defined by inequalities

$$D = \{(x,y,z) \in \mathbb{R}^3 \mid a \leq z \leq b,\ x^2 + y^2 \leq f^2(z)\}$$

in terms of one-variable integral involving the function $f(z)$.