Jacobs University
School of Engineering and Science Fall Term 2011

ESM 1B, Practice list

P.1. (Cosine theorem) Using dot products, prove that for any triangle ABC,

$$|BC|^2 = |AB|^2 + |AC|^2 - 2|AB| \cdot |AC| \cos(\angle BAC).$$

P.2. Find the distance between lines AB and CD, where

$$A = (0, 1, 2), \quad B = (3, 4, 5), \quad C = (1, 0, 1), \quad D = (0, 2, 0).$$

P.3. Find the distance between lines given in parametric form by the following equations:

$$\vec{r} = \vec{a} + \lambda\vec{b}, \quad \vec{r} = \vec{c} + \lambda\vec{d},$$

where

$$\vec{a} = (0, 0, 0), \quad \vec{b} = (1, 1, 1), \quad \vec{c} = (1, 0, 0), \quad \vec{d} = (1, 2, 3),$$

and λ is a parameter.

P.4. Find the equation of a plane parallel to lines $x = y = 2$ and $y = z = 3$ and containing the point $(4, 5, 6)$.

P.5. Consider two planes given by the following parameter equations:

$$\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c}, \quad \vec{r} = \vec{a}_1 + \lambda\vec{b}_1 + \mu\vec{c}_1,$$

where

$$\vec{a} = (0, 0, 0), \quad \vec{b} = (1, 0, 1), \quad \vec{c} = (0, -1, 0),$$
$$\vec{a}_1 = (1, 0, 0), \quad \vec{b}_1 = (0, 0, 1), \quad \vec{c}_1 = (1, 1, 1),$$

and λ, μ are parameters. Find coordinate equations of the line, at which these two planes intersect.

P.6. Consider two planes given by the following coordinate equations:

$$x + y + 2z = 3, \quad -x + 3y - z = 5.$$

Find any nonzero vector parallel to the line of intersection of these two planes.

P.7. Give a coordinate equation of a plane parallel to the plane

$$x + y + z = 0$$

and such that the distance between the two planes is 1.

P.8. Find the area of the triangle with vertices $(1, 0, 1)$, $(0, 1, 0)$ and $(0, 0, 1)$.

P.9. Consider two spheres given by the following equations

$$x^2 + y^2 + z^2 + x + 2y = 5, \quad x^2 + y^2 + z^2 - x - y - z = 1.$$

Find a coordinate equation of the plane containing the intersection of these two spheres.

P.10. Find the distance from the point $(0, 1, 1)$ to the plane given by the equation

$$x - y + z = 3.$$

P.11. Express the derivative of the function

$$f(t) = g(e^t, \sin t)$$

through the partial derivatives $\partial g/\partial x$ and $\partial g/\partial y$ of the function g.

P.12. Write the function $f(x, y) = x + y$ in polar coordinates. Compute the partial derivatives of this function with respect to the polar coordinates.

P.13. Determine whether there is a function $f(x, y)$ satisfying the following equation:

$$df = \sin y\, dx + \cos x\, dy.$$

P.14. Give an example of a function $f(x, y)$, whose directional derivative at $(0, 0)$ along $(1, 1)$ is 1 and whose directional derivative at the same point along $(1, -1)$ is -2.

P.15. Find all local minima of the function

$$f(x, y) = x^2 + y^2 + 2x + 3y.$$

P.16. Evaluate the directional derivative of the function $f(x, y) = \sqrt{x^2 + y^2}$ at point $(0, 0)$ in the direction $(1, 1)$.

P.17. Write the second order approximation of the following function at the point $(1, 2)$:

$$f(x, y) = \sin(x + y)e^{x-y}.$$

P.18. How many points of local minima has the function $f(x, y) = \sin^2(x) + \sin^2(y)$ in the rectangle $-5 \le x \le 5$, $-5 \le y \le 5$?

P.19. Find the total differential of the function $f(x, y, z) = g(x) + g(y) + g(z)$, where g is a function of one variable. Express the answer through the derivative g' of the function g.

P.20. Compute the total differential of the following function:

$$f(x, y, z) = \sin(x + e^{yz}).$$

P.21. Compute partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for the following function:

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}.$$

P.22. Suppose that $w = g(x/y, z/y)$ is a differentiable function of x/y and z/y. Show then that

$$x\partial wx + y\partial wy + z\partial wz = 0.$$

P.23. Consider the differential

$$\omega = y(1 + x - x^2)dx + x(x + 1)dy.$$

Find a function $g(x)$ such that $g(x)\omega$ is an exact differential.

P.24. Transform the equation

$$\frac{\partial^2\varphi}{\partial x^2} = \frac{\partial^2\varphi}{\partial y^2}$$

to new coordinates $s = \frac{1}{2}(x+y)$, $t = \frac{1}{2}(x-y)$. Show that φ has the form

$$f(x+y) + g(x-y)$$

for some functions f and g of one variable.

P.25. Find the second-order approximation of the following function at $(1,2)$:

$$f(x,y) = \exp(x^2+y^2).$$

P.26. Find all maxima, minima and saddle points of the function

$$f(x,y) = xy(x^2+y^2-1).$$

P.27. Can the following vector fields be represented as gradients of scalar fields:
(a) $\vec{A} = (y^2, x^2, z^2)$; (b) $\vec{A} = (-y, x, 0)$; (c) $\vec{A} = (y, x, z)$;
(d) $\vec{A} = (yz\cos(xy), xz\cos(xy), \sin(xy))$?
If no, explain. If yes, find the corresponding scalar fields.

P.28. Find the tangent plane to the surface

$$x = f(u+v), y = g(u-v), z = u$$

at point with coordinates $u = u_0$, $v = v_0$. Here f and g are some functions of one variable.

P.29. Find the tangent plane to the surface

$$f(xy) + g(y+z) = 1$$

at point $(1,1,1)$.

P.30. Parametrize the surface of the ellipsoid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

near the point $(a,0,0)$. Find the area form dA with respect to this parametrization.

P.31. Parametrize the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Write the length of the ellipse as a definite integral.

P.32. Let $\vec{r}(s)$ be a plane curve parametrized by the arc-length parameter. Denote by $\vec{v}(s)$ the velocity vector $\frac{d\vec{r}(s)}{ds}$, and by $\vec{n}(s)$ the unit vector

$$\frac{1}{\kappa(s)}\frac{d^2\vec{r}(s)}{ds^2},$$

where $\kappa(s)$ is the curvature at point $\vec{r}(s)$. Show that

$$\frac{d\vec{n}(s)}{ds} = -\kappa(s)\vec{v}(s).$$

P.33. Evaluate the integral

$$\iiint_C (x^2 + y^2 + \sin z)\, dx\, dy\, dz,$$

where C is the cylinder

$$x^2 + y^2 \leq 1, \quad -\pi \leq z \leq \pi.$$

P.34. Reduce the triple integral

$$\iiint_D f(x^2 + y^2 + z^2)\, dx\, dy\, dz$$

to a definite one-variable integral. Here D is the region given by the inequalities

$$1 \leq x^2 + y^2 + z^2 \leq 2,$$

and f is a function of one variable.

P.35. Find the surface area of the cylinder

$$x^2 + y^2 \leq a, \quad 0 \leq z \leq b.$$

P.36. Find the volume of the region

$$0 \leq x \leq 1, \quad 0 \leq y \leq x, \quad z \leq e^{x+y}.$$

P.37. Express the area of the surface

$$0 \leq x, y, \leq 1, \quad z = f(x, y)$$

as a double integral.

P.38. Find the tangent line to the curve

$$x = \cos(t), \quad y = \sin(2t)$$

at point $t = \pi/4$.

P.39. Parametrize the hyperbola $xy = 1$. Find the curvature as a function of the parameter.

P.40. Write an equation of the tangent line to the curve

$$x^3 + y^3 = 1$$

at point (x_0, y_0) of the curve.

P.41. Find the volume of the following region:

$$x^2 + y^2 \leq f(z).$$

Give the answer in terms of definite one-variable integral(s) involving the function f.

P.42. Find the surface area of the following region:

$$x^2 + y^2 \leq f(z).$$

Give the answer in terms of definite one-variable integral(s) involving the function f.

P.43. Evaluate the integral

$$\iint_S \vec{A} \cdot d\vec{S},$$

where S is the surface of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

and $\vec{A}$ is the vector field $(x^2, y, -z)$.

P.44. Evaluate the following integrals over the sphere $x^2 + y^2 + z^2 = 1$:

$$\iint_S (x \sin x, y \sin x, z \sin x) \cdot d\vec{S}.$$

P.45. Evaluate the integral

$$\iint_{x^2+y^4+z^6=1} (-2y^3, -3z^5 + z, 2y^3) \cdot d\vec{S}.$$

P.46. Consider the curve given by the following parametric equations:

$$x = \cos t, \quad y = \sin t, \quad z = \sin 2t, \quad 0 \le t \le 2\pi$$

Evaluate the integral

$$\int_C \vec{A} \cdot d\vec{r},$$

where $\vec{A} = (yze^{xy}, xze^{xy}, e^{xy})$ or $\vec{A} = (yz \cos(xyz) + x, xz \cos(xyz) + y, xy \cos(xyz) + z)$.

P.47. Evaluate the following integral

$$\int_C (xy^2 + 5x^2, y^3 + 5xy^2 + 7x) \cdot d\vec{r},$$

where C is the unit circle $x^2 + y^2 = 1$.

P.48. Consider the vector field

$$\vec{Q} = (3x^2(y + z) + y^3 + z^3, 3y^2(z + x) + z^3 + x^3, 3z^2(x + y) + x^3 + y^3).$$

Show that the integral $\int_L \vec{Q} \cdot d\vec{r}$ along a curve L connecting the points $(1, -1, 1)$ and $(2, 1, 2)$ does not depend on a particular choice of L. Compute this integral.

P.49. Find the area bounded by the following curves:

$$x^{2/5} + y^{2/5} = a^{2/5}, \quad x^{2/3} + y^{2/3} = a^{2/3}.$$

Here a is a constant.

P.50. Evaluate the integral

$$\int_C [y(4x^2 + y^2)dx + x(2x^2 + 3y^2)dy]$$

around the ellipse $x^2/a^2 + y^2/b^2 = 1$.

P.51. Consider the curve on the unit sphere $x^2 + y^2 + z^2 = 1$ given in spherical coordinates (φ, ξ) by the equation $\xi = 1 + \cos^2 \varphi$. Evaluate the integral

$$\oint_C (z^2/2, -y(x + z), x^2/2) \cdot d\vec{r}$$

over this curve.

P.52. An electric circuit contains a resistance R, a capacitor C and a battery supplying a time-varying electromotive force $V(t)$. The charge q on the capacitor therefore obeys the equation

$$R\frac{dq}{dt} + \frac{q}{C} = V(t).$$

Assuming that initially there is no charge on the capacitor, and given that $V(t) = V_0 \sin \omega t$, find the charge on the capacitor as a function of time.

P.53. Solve the following differential equations:

(a) $(y-x)y' + 2x + 3y = 0$; (b) $xy' + (x-1)y + x^2 = 0$; (c) $2xy' + y^2 = 1$;

(d) $y - y' = y^2 + xy'$; (e) $x^2y' = y(x+y)$; (f) $y'^3 - y'e^{2x} = 0$;

(g) $(1-x^2)\,dy + xy\,dx = 0$; (h) $y'^2 + 2(x-1)y' - 2y = 0$; (i) $x^2y' - 2xy = 3y$.

P.54. Find integrating factors for the following equations:

(a) $y' = \frac{1}{x-y^2}$; (b) $dy\left(x - \frac{2}{y}\right) - y\,dx = 0$; (c) $(1-x^2)y' + 2xy = (1-x^2)^{3/2}$;

(d) $y' - y\frac{\cos x}{\sin x} + \frac{1}{\sin x} = 0$; (e) $(x+y^3)y' = y$.

P.55. Find a plane curve, whose family of tangent lines is $y = 2ax + a^3$, where a is a real parameter.

P.56. Using the substitutions $u = x^2$, $v = y^2$, reduce the equation

$$xyy'^2 - (x^2 + y^2 - 1)y' + xy = 0$$

to Clairaut's form. Find the integral curves.

P.57. Solve the following initial value problems:

(a) $y' - (y/x) = 1, \quad y(1) = -1$; (b) $y' - y\tan x = 1, \quad y(\pi/4) = 3$.

P.58. Find general solutions to the following second-order linear ODEs:

(a) $y'' - 5y' + 6y = \sin x$; (b) $y'' + 2y' + 6 = e^{2x}$;

(c) $y'' - 2y' + y = e^{2x}$; (d) $y'' + y = \cos x$;

P.59. A simple harmonic oscillator of mass m and natural frequency ω_0 experiences an oscillating driving force $f(t) = ma \cos \omega t$. The equation of motion is

$$\ddot{x} + \omega_0^2 x = a \cos \omega t.$$

Here x is the position. Given that $x = \dot{x} = 0$ at $t = 0$, find x as a function of t.

P.60. Solve the following initial value problems:

(a) $y'' + 2y' + 5y = 0, \quad y(0) = 1, \quad y'(0) = 0$;

(b) $y'' + 2y' + 5y = e^{-x}\cos 3x, \quad y(0) = 0, \quad y'(0) = 0$.

P.61. A solution of the differential equation

$$y'' + 2y' + y = 4e^{-x}$$

takes the value 1 when $x = 0$ and e^{-1} when $x = 1$. What is its value when $x = 2$?

P.62. Two functions $x(t)$ and $y(t)$ satisfy the following system of ODEs:

$$\frac{dx}{dt} - 2y = -\sin t, \qquad \frac{dy}{dt} + 2x = 5\cos t.$$

Find $x(t)$ and $y(t)$ assuming that $x(0) = 3$ and $y(0) = 2$.

P.63. Find general solutions of the following differential equations:

(a) $y'' - y = x^n$; (b) $y'' - 2y' + y = 2xe^x$.

P.64. Find an explicit expression for a sequence u_n satisfying

$$u_{n+1} + 5u_n + 6u_{n-1} = 2^n, \quad u_0 = u_1 = 1.$$