Jacobs University
School of Engineering and Science

Fall Term 2011

Introductory Complex Analysis, Homework 1

Due Date: Friday, September 16, in class.

Problems marked ($\star$) are bonus ones.

1.1. Denote by i_C the inversion in circle C. Show that i_C takes lines and circles to lines and circles.

1.2. Let C be the circle $|z| = 1$. Draw the image $i_C(P)$, where P is

(a) the line $\operatorname{Re} z = \frac{1}{2}$;

(b) the square with vertices $(-1, 1), (1, -1), (-1, -3), (-3, -1)$;

(c) the upper half-plane $\operatorname{Im} z > 0$.

1.3. Show that

(a) for any pair of non-intersecting circles C_1 and C_2 there exists a circle C such that i_C takes C_1 and C_2 to concentric circles;

(b) for any pair of circles C_1 and C_2 there exists a circle C such that i_C takes C_1 to C_2.

1.4. Two points $z, z' \in \mathbb{C}$ are *symmetric* with respect to a circle C if $i_C(z) = z'$. Show that any circle C' through a pair of symmetric with respect to C points z, z' is orthogonal to C.

1.5. (a) Show that any Möbius map can be represented as a composition of transformations $z \to \frac{1}{z}$, $z \to z + a$ and $z \to bz$ for some $a, b \in \mathbb{C}$;

(b) show that Möbius map is a bijection from $\widehat{\mathbb{C}}$ to $\widehat{\mathbb{C}}$;

(c) show that Möbius maps form a group;

(d) show that any Möbius transformation takes lines and circles to lines and circles.

1.6. (a) Let (u_1, u_2, u_3) be a triple of points of $\widehat{\mathbb{C}}$. Find a Möbius map g, such that $g(u_1) = 0$, $g(u_2) = 1$, $g(u_3) = \infty$.

(b) Show that if a Möbius map fixes at least three non-collinear points then it is the identity map.

1.7. (a) Find all Möbius maps that preserve the real axis;

(b) Find all Möbius maps that preserve the unit disk $|z| < 1$.

(c) Find all Möbius maps that take the upper half-plane $\operatorname{Im} z > 0$ to the unit disk $|z| \leq 1$.

1.8. ($\star$) Show that Möbius transformations preserve *cross ratio*

$$[z_1, z_2, z_3, z_4] = \frac{z_3 - z_1}{z_3 - z_2} : \frac{z_4 - z_1}{z_4 - z_2}$$

of four points.

1.9. ($\star$) Show that inversions (and Möbius transformations) preserve angles between lines and circles.

1.10. A function $f : D \to \mathbb{C}$ is *antiholomorphic* if f is $\mathbb{R}$-differentiable in D and $\frac{\partial f}{\partial z} = 0$.
Show that $f = u + iv$ is antiholomorphic if and only if $\bar{f} = u - iv$ is holomorphic.

1.11. Show that a composition of holomorphic functions is holomorphic.

1.12. Let f be holomorphic. Show that

(a) if $|f(z)|$ is constant then f is constant;

(b) if $f(z)$ is real for every $z \in D$ then f is constant.

1.13. Show that $f(z)$ is holomorphic in D if and only if $\overline{f(\bar{z})}$ is holomorphic in $\overline{D}$.

1.14. Show that a harmonic function u satisfies the equation

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0$$