Jacobs University
School of Engineering and Science

Fall Term 2011

Introductory Complex Analysis, Homework 2

Due Date: Friday, September 30, in class.

Problems marked $(\star)$ are bonus ones.

2.1. Determine the radii of convergence of the following power series:

(a) $\sum_{n=0}^{\infty} n^p z^n$; (b) $\sum_{n=0}^{\infty} n! z^n$; (c) $\sum_{n=1}^{\infty} (n+i) z^n$; (d) $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} z^n$; (e) $\sum_{n=1}^{\infty} \left(\frac{z}{\ln z}\right)^n$; (f) $\sum_{n=1}^{\infty} \frac{z^{2n-1}}{2n-1}$; (g) $\sum_{n=0}^{\infty} z^{n!}$.

2.2. For which $z \in \mathbb{C}$ the following series converge?

(a) $\sum_{n=1}^{\infty} \left(\frac{z}{1+z}\right)^n$; (b) $\sum_{n=1}^{\infty} \frac{z^n}{1+z^{2n}}$.

2.3. $(\star)$ (a) Consider a complex polynomial

$$p(z) = \sum_{k=0}^{d} a_k z^k$$

as a power series expansion around the origin. Express p as a power series around some point $w \neq 0$.

(b) Let f be a rational function

$$f(z) = \frac{3z^4 + z^3 + 2z^2 + 7}{z(z-1)}$$

Express f explicitly as a power series around $z = 2$. Find the convergency radius.

(c) Write down an algorithm to express as a power series an arbitrary rational map $f = p/q$ with two polynomials p and q, around any point $w \in \mathbb{C}$ with $q(w) \neq 0$.

2.4. Show equivalence of two definitions of the exponent:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad \text{and} \qquad e^z = \lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n$$

2.5. Define hyperbolic trigonometric functions

$$\sinh z = \frac{e^z - e^{-z}}{2} \qquad \text{and} \qquad \cosh z = \frac{e^z + e^{-z}}{2}$$

(a) Express them through $\cos iz$ and $\sin iz$.

(b) Express $|\cos z|^2$ and $|\sin z|^2$ in terms of trigonometric functions of x and y (for $z = x + iy$).

2.6. Show that $\sin^2 z + \cos^2 z = 1$ and $\cos 2z = 2\cos^2 z - 1$.

2.7. For every of the functions $\sin z$, $\cos z$, $\tan z$ find all the points $z \in \mathbb{C}$ where the functions are

(a) real;

(b) purely imaginary.

2.8. For functions $f(z) = \sin z,\ \tan z,\ \cot z$ find their periods and maximal domains on which f is injective.

2.9. Find the image of horizontal line $y = b$ and vertical line $x = a$ under the map

(a) $z \to e^z$;

(b) $z \to \cos z$

for every $a, b \in \mathbb{R}$.