Jacobs University
School of Engineering and Science

Fall Term 2011

# Introductory Complex Analysis, Homework 3

**Due Date:** Friday, October 14, in class.

Problems marked $(\star)$ are bonus ones.

**3.1.** Construct explicitely a function mapping conformally and bijectively

(a) the angle $\{z \in \mathbb{C} \,|\, -\pi/3 < z < \pi/3\}$ onto the upper halfplane;

(b) the half-disk $\{z \in \mathbb{C} \,|\, |z| < 1,\ \operatorname{Im} z > 0\}$ onto the unit disk;

(c) the unit disk onto the strip $\{z \in \mathbb{C} \,|\, 0 < \operatorname{Im} z < 1\}$;

(d) the domain $\{z \in \mathbb{C} \,|\, |z| > 1,\ \operatorname{Im} z > 0,\ \operatorname{Re} z > 0\}$ onto the positive quadrant $\{z \in \mathbb{C} \,|\, \operatorname{Im} z > 0,\ \operatorname{Re} z > 0\}$;

(e)$(\star)$ the domain $\{z \in \mathbb{C} \,|\, |z| < 1,\ \operatorname{Im} z > 0,\ z \notin [i/2, i]\}$ onto the upper halfplane;

**3.2.** Let $f : D \to \mathbb{C}$ be holomorphic and bijective onto the image $f(D)$. Show that $f^{-1} : f(D) \to D$ is also holomorphic.

**3.3.** Let $f(z)$ be holomorphic in domain $D$, and $z \notin \bar{D}$. Show that

$$\frac{1}{2\pi i} \int_{\partial D} \frac{f(\xi)}{\xi - z}\, d\xi = 0$$

**3.4.** Compute the following integrals:

$$(a) \int_{[0,1+i]} y\, dz; \qquad (b) \int_{|z|=1} y\, dz.$$

**3.5.** Compute the following integrals:

$$(a) \int_{|z|=2} \frac{dz}{z^2 - 1}; \qquad (b) \int_{|z-3i|=3} \frac{dz}{z^2 + 1}; \qquad (c) \int_{|z-1-i|=2} \frac{dz}{(z-1)^2(z^2+1)};$$

**3.6.** Let $D$ be a disk, and let $u : D \to \mathbb{R}$ be a harmonic function.

(a) Show that there exists a unique (up to constant) holomorphic in $D$ function $f$ with $\operatorname{Re} f = u$.

(b) Let $z \in D$, and the disk $|\xi - z| \le r$ is contained in $D$. Prove the *Mean Value Theorem* for harmonic functions:

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{it})\, dt$$

**3.7.** $(\star)$ Let $D$ be a disk in $\mathbb{C}$, $f$ and $g$ are holomorphic functions in $D$, and

$$f^2 + g^2 = 1$$

Prove that there exists a holomorphic function $h(z)$ such that

$$f(z) = \cos h(z), \qquad g(z) = \sin h(z)$$