Jacobs University
School of Engineering and Science

Fall Term 2011

Introductory Complex Analysis, Homework 4

Due Date: Friday, October 28, in class.

Problems marked ($\star$) are bone ones.

4.1. Let $P_1, \ldots, P_n$ be points lying in the exterior of a circle centered at O. Show that there is a point P on the circle such that the product of distances from P to $P_1, \ldots, P_n$ is greater than $OP_1 \cdot \ldots \cdot OP_n$, as well as point P' for which this product is less than $OP_1 \cdot \ldots \cdot OP_n$.

4.2. Let D be an open subset of $\mathbb{C}$, and suppose $f, g : D \to \mathbb{C}$ are holomorphic and $fg \equiv 0$ on D. Does it follow that $f \equiv 0$ or $g \equiv 0$ on D? If yes: prove it; if no: give a counterexample and state and prove a condition on D under which this is true.

4.3. (a) Suppose $f(z)$ has an isolated singularity at z_0. What kind of isolated singularity can $\exp(f(z))$ have at z_0?

(b) Let z_0 be an essential singularity of f. What is the type of singularity of $1/f$ at z_0?

4.4. For the following functions, determine all their isolated singularities (including at the point ∞) and their types. For a removable singularity determine what value can be "filled in", and for a pole, determine its order:

(a) $1/(1-e^z)$; (b) $\sin[\pi/(1+z^2)]$; (c) $\cos(1/z) + e^{3z}$; (d) $(z^2 - \sin(z^2))/z^5$.

4.5. Find maximal R such that the function $f(z) = z/\sin z$ in $|z| < R$ can be represented as a sum of its Taylor series centered at 0.

4.6. Show the following properties of entire function f:

(a) if f is not constant then $\lim_{r\to\infty} \max\limits_{|z|=r} |f(z)| = +\infty$;

(b) if $\max\limits_{|z|=r} |f(z)| \leq Ar^n + B$ for some constants A, B, N and any $r > 0$, then f is a polynomial of degree at most N;

(c) if $\mathrm{Re} f(z) \geq 0$ for all $z \in \mathbb{C}$, the f is constant;

(d) if $\lim\limits_{z\to\infty} f(z) = \infty$ then the set $\{z \in \mathbb{C} \mid f(z) = 0\}$ is not empty.

(e) prove *Fundamental Theorem of Algebra*: every polynomial has a root in $\mathbb{C}$.

4.7. The expression

$$\{f, z\} = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

is called the *Schwarzian derivative* of f. Let f have a multiple zero or pole at z_0. Compute the leading term of the the Laurent series of $\{f, z\}$ centered at z_0.

4.8. ($\star$) Show that

$$\int_0^\infty \cos x^2 \, dx = \sqrt{\pi/8} = \int_0^\infty \sin x^2 \, dx$$

Hint: integrate along the boundary of a domain with vertices 0, R, and $Re^{i\pi/4}$, and use the fact that $\int_0^\infty e^{-x^2} = \sqrt{\pi}/2$.