Jacobs University
School of Engineering and Science

Fall Term 2011

Introductory Complex Analysis, Homework 5

Due Date: Friday, November 11, in class.

Problems marked ($\star$) are bonus ones.

5.1. Define $f(z) = 1 + z^2 + z^4 + z^8 + \cdots = 1 + \sum_{n=1}^{\infty} z^{2^n}$.

(a) Show that f is holomorphic in the unit disk $|z| < 1$;

(b) show that $f(z) \to \infty$ as z approaches any point of type $e^{2\pi i/2^n}$;

(c) show that every point of the circle $|z| = 1$ is singular.

5.2. Let radius of convergence of a power series $f(z) = \sum c_n (z - z_0)^n$ be equal to R, and assume that the function $f(z)$ has a pole at some w_0 with $|w_0 - z_0| = R$. Show that for every w, $|w - z_0| = R$, the series $\sum c_n (w - z_0)^n$ does not converge absolutely.

5.3. Find poles and compute residues of the following functions:

(a) $\dfrac{1}{z^2 + 5z + 6}$; (b) $\dfrac{1}{(z^2 - 1)^2}$; (c) $\dfrac{1}{\sin^2 z}$; (d) $\cot z$; (e) $\dfrac{1}{z^m (1 - z)^n}$ $(m, n \in \mathbb{N})$.

5.4. Evaluate the following integrals:

(a) $\displaystyle\int_0^{\pi/2} \frac{dx}{a + \cos^2 x}$, $|a| > 1$; (b) $\displaystyle\int_0^{\infty} \frac{x^2\, dx}{x^4 + 5x^2 + 6}$; (c) $\displaystyle\int_0^{\infty} \frac{x \sin x\, dx}{x^2 + a^2}$, $a \in \mathbb{R}$.

5.5. ($\star$) (a) Show that $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{\cosh x} = \pi$

Hint: integrate over the boundary of rectangle with vertices $\pm R$, $\pm R + i\pi$.

(b) Compute $\displaystyle\int_0^{\infty} \frac{x^\alpha\, dx}{x^2 + 1}$, where $0 < \alpha < 1$.

Danger: z^α is not single-valued!

5.6. (a) How many roots of the polynomial $z^7 - 2z^5 + 6z^3 - z + 1$ lie in the unit disk?

(b) How many roots of the polynomial $z^4 + 10z + 1$ have their modulud between 1 and 2?

(c) How many roots of the polynomial $z^4 + 8z^3 + 3z^2 + 8z + 3$ have positive real part?

5.7. Let f be holomorphic in the unit disk and continuous in its closure. Show that

$$|f(0)| \le \frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})|\, dt$$

Hint: recall the mean value theorem.

5.8. ($\star$) Let f, g, h be holomorphic non-constant in some domain D. Show that the function

$$F(z) = |f(z)| + |g(z)| + |h(z)|$$

has no local maximum in D.

5.9. Let M be the set of all holomorphic maps f from the unit disk to itself (not obligatory surjective) satisfying $f(0) = 2/3$. Find $\sup_{f \in M} |f'(0)|$. *Hint:* use Schwarz Lemma.