Jacobs University
School of Engineering and Science
Fall Term 2011

Introductory Complex Analysis, Homework 6

Due Date: Friday, November 25, in class.

Problems marked ($\star$) are bonus ones.

6.1. Let $\{f\}$ be a locally uniformly bounded family of holomorphic functions on D. Show that $\{f'\}$ is also locally uniformly bounded.

6.2. Show that the functional J on the set of holomorphic functions defined by $J(f) = f^{(p)}(z_0)$ is continuous.

6.3. Show that the series

$$\sum_{n=0}^{\infty} \frac{1}{z^n + z^{-n}}$$

converges for $|z| < 1$ and for $|z| > 1$ to two distinct holomorphic functions.

6.4. *Riemann Mapping Theorem.*
Let $D \subset \mathbb{C}$ be a simply-connected domain which is not the whole plane. Then there exists a conformal map from D to the unit disk Δ. The map is unique up to Mobius transformation of Δ.

First, let us prove that there exist univalent functions from D to Δ.

(a) Let $a, b \notin D$, $a, b \in \hat{\mathbb{C}}$. Define function $f(z) = \sqrt{\frac{z-a}{z-b}}$. Show that two germs $(U_1, f_1), (U_2, f_2)$ of f have analytic continuations f_1, f_2 in D, $f_1 = -f_2$, with f_1, f_2 being univalent, and $f_1(D) \cap f_2(D) = \emptyset$.

(b) Let $W = \{w \in \mathbb{C} \,|\, |w - w_0| \leq r\}$ be any disk contained in $f_2(D)$ (Open Mapping Theorem). Show that the function $\tilde{f} = \frac{r}{f_1(z) - w_0}$ is holomorphic in D, and $|\tilde{f}(z)| < 1$ for any $z \in D$.

(c) Fix $z_0 \in D$. Show that the family of holomorphic functions $\{f\} = \{f : D \to \Delta \,|\, f \text{ is univalent, } f(z_0) = 0\}$ is non-empty.

Now find the required function.

(d) Show that $\{f\}$ is normal.

(e) Show that there is function $f_0 \in \{f\}$ such that $0 < |f_0'(z_0)| \geq |g'(z_0)|$ for every $g \in \{f\}$.

Finally, prove that $f_0(D) = \Delta$. Assume that $c \in \Delta \setminus f_0(D)$.

(f) Show that a germ of function $g(z) = \sqrt{\frac{f_0(z) - c}{1 - \bar{c} f_0(z)}}$ has an analytic continuation g_1 in D.

(g) Show that the function $h(z) = \frac{g_1(z) - g_1(z_0)}{1 - \overline{g_1(z_0)} g_1(z)}$ belongs to $\{f\}$, and $|f_0'(z_0)| < |h'(z_0)|$.

6.5. Euler *Gamma function* is defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, dt$$

(a) Show that functions $F_n(z) = \int_{1/n}^{\infty} e^{-t} t^{z-1} \, dt$ are holomorphic in $\mathbb{C}$.

(b) Show that $\Gamma(z)$ is holomorphic in the right halfplane $\mathrm{Re} z > 0$.

(c) Show that for $\mathrm{Re} z > 0$ Gamma function satisfies the equation $\Gamma(z+1) = z\Gamma(z)$.

(d) Prove that $\Gamma(z)$ has an analytic continuation to $\mathbb{C} \setminus \{0, -1, -2, \dots\}$.